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Mathematical Physics II

Frobenius Method and Special Functions:

Singular Point:

If a differential equation at any certain point may have not exist, then it is said that the differential equation posses a singularity at that point. For an example let us consider a differential equation,

$$y'' + \frac{1}{x^3}y = 0$$

Here the above equation posses a singularity at $x = 0$. But this is not always true because it can be shown that the differential equation,

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0$$

posses a singularity at $x = 0$, though yet it has a unique series solution. Thus there are some conditions which imposes limitation in obtaining the series solution. This is given by Fuch's theorem.
To explain Fuch's theorem, we first consider a second-order homogeneous differential equation,

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

Now if you want to define ordinary and singular points, then we can proceed as, if the functions $p(x)$ and $q(x)$ remain finite at $x = x_0$, point $x = x_0$ is an ordinary point. However, if either $p(x)$ or $q(x)$ (or both) diverges as $x \to x_0$, point x_0 is a singular point. If either $p(x)$ or $q(x)$ diverges as $x \to x_0$ but $(x-x_0)p(x)$ and $(x - x_0)q(x)$ remain finite, i.e. $\lim_{x\to x_0}(x - x_0)p(x) \to finite$ and $\lim_{x\to x_0}(x - x_0)^2q(x) \to finite$ as $x \to x_0$, then $x = x_0$ is called a regular, or nonessential, or removable singular point. If $p(x)$ diverges faster than $\frac{1}{(x-x_0)}$ so that $(x - x_0)p(x)$ goes to infinity as $x \to x_0$, or $q(x)$ diverges faster than $\frac{1}{(x-x_0)^2}$ so that $(x - x_0)^2q(x)$ goes to infinity as $x \to x_0$, then point $x \to x_0$ is labeled an irregular, or essential, or non-removable singularity.
This concept has its usefulness in (1) classifying ordinary differential equations (ODEs) and (2) investigating the feasibility of a series solution.
Example:
Legendre's equation has the form,

$$(1 - z^2)y'' - 2zy' + l(l + 1)y = 0 \tag{2}$$

where, l is a constant. Firstly, divide through by $(1 - z^2)$ to put the equation into our standard form like equation (1) we therefore get,

$$y'' - \frac{2z}{1-z^2}y' + \frac{l(l+1)}{1-z^2}y = 0$$

Here we get $p(z)$ and $q(z)$ as

$$p(z) = \frac{-2z}{1-z^2} = \frac{-2z}{(1+z)(1-z)}$$

,

$$q(z) = \frac{l(l+1)}{(1+z)(1-z)}$$

By inspection, $p(z)$ and $q(z)$ are analytic at $z = 0$, which is therefore an ordinary point, but both diverge for $z = \pm 1$, which are therefore singular points. However, at $z = 1$ we see that both $(z-1)p(z)$ and $(z-1)^2 q(z)$ are analytic and hence $z = 1$ is a regular singular point. Similarly, at $z = -1$ both $(z+1)p(z)$ and $(z+1)^2 q(z)$ are analytic, and it too is a regular singular point.

So far we have assumed that z_0 is finite. However, we may sometimes wish to determine the nature of the point $|z| \to \infty$. This may be achieved straightforwardly by substituting $\omega = \frac{1}{z}$ into the equation and investigating the behavior at $\omega = 0$.

Exercise:

1. Show that Laguerre's equation has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$.
2. Show that Bessel's equation has a regular singularity at $x = 0$ and an irregular singularity at $x = \infty$.

Inexact or Series Solution of Differential Equation:

The Second order differential equations of all types are not always find to have solution in terms of exact function. We developed methods for solving some equations in which the coefficients were not constant but functions of the independent variable x. In that case we were able to write the solutions to such equations in terms of elementary functions, or as integrals. However, the solutions of equations with variable coefficients cannot be written in the said way, therefore we must consider alternative approaches. Here we discuss a method for obtaining inexact solutions to linear ODEs in the form of convergent series. Such series can be evaluated numerically, and those occurring most commonly are named and tabulated. To discuss the procedure let us consider a second order linear ODE of type,

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

Let the above equation does not posses a solution, which can not be expressed as an exact function, $y = f(x)$ or $\phi(x)$. In this case we assume,

$$y = \sum_{l=0}^{\infty} a_l x^{k+l} = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + ...\infty \tag{4}$$

Substituting this solution in the above equation (3) we can obtain a certain number of conditions by which the co-efficients a_0, a_1, a_2 etc. can be determined. This method of obtaining the series solution is known as Frobenius' Method.

Let us consider,

$$y'' + xy' + y = 0 \tag{5}$$

let,

$$y = \sum_{l=0}^{\infty} a_l x^{k+l} \tag{6}$$

Therefore,

$$y' = \sum_{l=0}^{\infty} a_l (k+l) x^{k+l-1} \tag{7}$$

and

$$y'' = \sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2} \tag{8}$$

Therefore we get from equation (5),

$$\sum_{l=0}^{\infty} a_l (k+l)(k+l-1) x^{k+l-2} + \sum_{l=0}^{\infty} a_l (k+l) x^{k+l} + \sum_{l=0}^{\infty} a_l x^{k+l} \tag{9}$$

This equation (9) contains an infinite numbers of terms which is completely arbitrary and since their sum is zero, so we can conclude that the individual terms must be equals to zero.
Equating the co-efficient of lowest power of x to be zero we get,

$$a_0 k(k-1) = 0 \tag{10}$$

This equation (10) is known as Identical equation. If we now assume $a_0 \neq 0$, then either $k = 0$ or $k = 1$.
Equating the co-efficient of next higher power of x to be zero, we therefore get,

$$a_1 k(k+1) = 0 \tag{11}$$

If $k = 0$, then $a_1 \neq 0$, but if $k = 1$, then $a_1 = 0$. Equating the co-efficient of x^{k+j}th term to be zero we get,

$$a_{j+2}(k+j+2)(k+j+1) + a_j(k+j) + a_j = 0 \tag{12}$$

Therefore,

$$a_{j+2} = \frac{-a_j}{k+j+2} \tag{13}$$

This is called Recurrence Relation. From this relation we can obtain all the co-efficients which gives us the complete solution of y of the differential equation.

Case-I:

When $k = 0$,

$$a_{j+2} = \frac{-a_j}{j+2} \tag{14}$$

$a_2 = \frac{-a_0}{2}$, $a_3 = \frac{-a_1}{3}$, $a_4 = \frac{-a_2}{4} = \frac{a_0}{2.4}$, $a_5 = \frac{-a_3}{5} = \frac{a_1}{3.5}$, $a_6 = \frac{-a_0}{2.4.6}$, $a_7 = \frac{-a_1}{3.5.7}$
Therefore the solution,

$$y = \sum_{l=0}^{\infty} a_l x^l = a_0\left(1 - \frac{1}{2}x^2 + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + ...\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - ...\right) \tag{15}$$

Case-II:

When $k=1$,

$$a_{j+2} = \frac{-a_j}{j+3} \tag{16}$$

$a_2 = \frac{-a_0}{3}$, $a_4 = \frac{-a_2}{5} = \frac{a_0}{3.5}$, $a_6 = \frac{-a_0}{3.5.7}$
Therefore the solution becomes,

$$y = \sum_{l=0}^{\infty} a_l x^{1+l} = a_0(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + ...) \tag{17}$$

Here we note that the solution (17) is just a part of solution (15), therefore these two part are not linearly independent. Hence the two linearly independent solution of the given equation is

$$y_1 = \sum_{l=0}^{\infty} a_l x^l \tag{18}$$

and

$$y_2 = \sum_{l=0}^{\infty} a_l x^{1+l} \tag{19}$$

Series Solution of Different Type of Second Order Differential Equations:

Second order differential equations possessing Fuchsian type of singularity always have a unique series solution. To determine the series solutions it has been found that there may be three cases which are-
(i) Roots of the indicial equation are distinct and differ by an integer,
(ii) Roots of the indicial equation are distinct and do not differ by an integer,
(iii) Roots of the indicial equation are equal
Here in the first two cases we can obtain two distinct solutions by putting the distinct values of k but in the third case when both the roots are equal (or, identical) the two solutions are given by,

$$y_1 = [y(k)]_{k=k_1} \tag{20}$$

and

$$y_2 = [\frac{\partial y}{\partial k}]_{k=k_1} \tag{21}$$

Therefore the general solution is given by,

$$y = A[y(k)]_{k=k_1} + B[\frac{\partial y}{\partial k}]_{k=k_1} \tag{22}$$

Exercise:
Find the series solution of the following equations-
1.

$$(2x + x^3)y'' - y' - 6xy = 0$$

2.

$$2x^2y'' - xy' + (x^2+1)y = 0$$

3.

$$4x^2y'' + y = 0$$

Power Series Method:

Apart from Frobenius method for solving the ODEs with variable coefficients, there is another method, called the power series method because it gives solutions in the form of a power series, $a_0 + a_1x + a_2x^2 + a_3x^3 + ...$ These series can be used for computing values, graphing curves, proving formulas, and exploring properties of solutions, as we shall see.
From calculus we remember that a power series (in powers of $x - x_0$) is an infinite series of the form-

$$\sum_{m=0}^{\infty} a_m(x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + ... \tag{23}$$

Here, x is a variable. $a_0, a_1, a_2...$ are constants, called the coefficients of the series. x_0 is a constant, called the center of the series. In particular, if $x_0 = 0$, we obtain a power series in powers of x.

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + a_3x^3... \tag{24}$$

We shall assume that all variables and constants are real. We note that the term "power series" usually refers to a series of the form (23) or (24) but does not include series of negative or fractional powers of x. We use m as the summation letter, reserving n as a standard notation in the Legendre and Bessel equations for integer values of the parameter.
The idea of the power series method for solving linear ODEs seems natural, once we know that the most important ODEs in applied mathematics have solutions of this form. We explain the idea by an ODE that can readily be solved otherwise.
Example:
Let us solve a linear, homogeneous, ODE, $y' - y = 0$.
In the first step we insert,

$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + a_3x^3... \tag{25}$$

and the series obtained by term wise differentiation,

$$y' = \sum_{m=0}^{\infty} m a_m x^{m-1} = a_1 + 2a_2x + 3a_3x^2 + ... \tag{26}$$

If we substitute these values in the above ODE, therefore we get,

$$(a_1 + 2a_2x + 3a_3x^2 + ...) - (a_0 + a_1x + a_2x^2 + ...) = 0 \tag{27}$$

Then we collect like powers of x, finding

$$(a_1 - a_0) + (2a_2 - a_1)x + (3a_3 - a_2)x^2 + ... = 0 \tag{28}$$

Equating the coefficient of each power of x to be zero, we have $a_1 - a_0 = 0$, $2a_2 - a_1 = 0$, $3a_3 - a_2 = 0, ...$ Solving these equations, we may express $a_1, a_2...$ in terms of a_0, which remains arbitrary: $a_1 = a_0$, $a_2 = \frac{a_1}{2} = \frac{a_0}{2}$, $a_3 = \frac{a_2}{3} = \frac{a_0}{2.3}, ...$
With these values of the coefficients, the series solution becomes the familiar general solution-

$$y = a_0 + a_0x + \frac{a_0}{2}x^2 + \frac{a_0}{2.3}x^3... = a_0(1 + x + \frac{x^2}{2} + \frac{x^3}{2.3}...) = a_0 \exp x \tag{29}$$

Exercise:

1. Solve $y'' + y = 0$ by power series method.

Existence of Power Series Solutions:

Let us consider an ODE is in standard form-

$$y'' + p(x)y' + q(x)y = r(x) \tag{30}$$

If p, q and r in (30) are analytic at $x = x_0$ then every solution of (30) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $(x - x_0)$ with radius of convergence > 0.

Bessel Functions:

One of the most important ODEs in Physics and Mathematics is Bessel's equation. The differential equation of the form,

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \tag{31}$$

is known as Bessel's differential equation. Such kind of equation arises in the separation of wave equation in cylindrical polar co-ordinate system or the separation of Helmholtz equation in the spherical polar coordinate system. The above equation has a regular singular point at $x = 0$ and an essential singularity at $x = \infty$. The parameter n is a given number, which we may take as ≥ 0 with no loss of generality.
[Fuch's condition for the differential equation, $x^2y''+xy'+(x^2-n^2)y = 0$***, is*** $\lim_{x\to 0}(x-0)\frac{1}{x} \to 1$***, i.e. a finite value and*** $\lim_{x\to 0}(x - 0)^2(1 - \frac{n^2}{x^2}) \to -n^2$***, this is also a finite value. Since the above equation contains a Fuchsian type of singularity at*** $x = 0$***, therefore we can obtain a series solution of this type of equation by Frobenius method.]***
Let us consider,

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \tag{32}$$

Therefore,

$$y' = \sum_{r=0}^{\infty} a_r (k+r) x^{k+r-1} \tag{33}$$

and

$$y'' = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r-2} \tag{34}$$

Therefore, by substitute y' and y'' we get from equation (31),

$$\sum_{r=0}^{\infty} a_r (k+r)(k+r-1) x^{k+r} + \sum_{r=0}^{\infty} a_r (k+r) x^{k+r} + \sum_{r=0}^{\infty} a_r x^{k+r+2} - n^2 \sum_{r=0}^{\infty} a_r x^{k+r} = 0 \tag{35}$$

Since the series contains infinite number of terms so that the co-efficient of each term should be zero individually. Now equating the lowest power of x to be zero we get,

$$a_0 k(k-1) + a_0 k - n^2 a_0 = 0 \tag{36}$$

or,

$$a_0(k^2 - n^2) = 0 \tag{37}$$

as $a_0 \neq 0$, therefore we must have $k = \pm n$
Equating the co-efficient of x^{k+1} to be zero,

$$a_1 k(k+1) + a_1(k+1) - n^2 a_1 = 0 \tag{38}$$

or,

$$a_1[(k+1)^2 - n^2] = 0 \tag{39}$$

as$[(k+1)^2 - n^2] \neq 0$, therefore $a_1 = 0$. Now, equating the co-efficient of x^{k+j}th term to be zero,

$$a_j(k+j)(k+j-1) + a_j(k+j) + a_{j-2} - n^2 a_j \tag{40}$$

or,

$$a_j = \frac{-a_{j-2}}{(k+j)^2 - n^2} \tag{41}$$

Therefore,

$$a_2 = \frac{-a_0}{(k+2)^2 - n^2} \tag{42}$$

$$a_3 = 0 \tag{43}$$

$$a_4 = \frac{-a_2}{(k+4)^2 - n^2} \tag{44}$$

$$a_5 = 0 \tag{45}$$

$$a_6 = \frac{-a_4}{(k+6)^2 - n^2} \tag{46}$$

Now for $k = +n$ we get,

$$a_2 = \frac{-a_0}{2(2n+2)} \tag{47}$$

$$a_4 = \frac{a_0}{2(2n+2)4(2n+4)} \tag{48}$$

$$a_6 = \frac{-a_0}{2.4.6(2n+2)(2n+4)(2n+6)} \tag{49}$$

Therefore the solution,

$$y_1 = a_0[x^n - \frac{x^{n+2}}{2(2n+2)} + \frac{x^{n+4}}{2.4(2n+2)(2n+4)} - \frac{x^{n+6}}{2.4.6(2n+2)(2n+4)(2n+6)} + ...] \tag{50}$$

Let,

$$a_0 = \frac{1}{2^n\Gamma(n+1)} \tag{51}$$

Therefore we can write,

$$y_1 = [\frac{(\frac{x}{2})^n}{\Gamma(n+1)} - \frac{(\frac{x}{2})^{n+2}}{(n+1)\Gamma(n+1)} + \frac{(\frac{x}{2})^{n+4}}{2.(n+1)(n+2)\Gamma(n+1)} - \frac{(\frac{x}{2})^{n+6}}{2.3.(n+1)(n+2)(n+3)\Gamma(n+1)} + ...] \tag{52}$$

or,

$$y_1 = [\frac{(\frac{x}{2})^n}{\Gamma(n+1)} - \frac{(\frac{x}{2})^{n+2}}{\Gamma(n+2)} + \frac{(\frac{x}{2})^{n+4}}{2\Gamma(n+3)} - \frac{(\frac{x}{2})^{n+6}}{3!\Gamma(n+4)} + ...] \tag{53}$$

Therefore,

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(\frac{x}{2})^{n+2s}}{s!\Gamma(n+s+1)} \tag{54}$$

Here the above series solution gives us a polynomial of x and this solution of the Bessel's equation is known as Bessel's polynomial of 1st kind and it is denoted by $J_n(x)$.
Similarly by putting $k = -n$, we get another solution of the Bessel's differential equation which is,

$$J_{-n}(x) = \sum_{s=n}^{\infty} \frac{(-1)^s(\frac{x}{2})^{-n+2s}}{s!\Gamma(-n+s+1)} \tag{55}$$

for $n =$ integer, but $s = o \to \infty$ for $n =$ non-integer.
Here it is to noted that if n is an integer, then $J_n(x)$ and $J_{-n}(x)$ are not independent solutions. But if n is an non-integer, then however $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of Bessel's equation.