Online Course Materials :

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17(b) show that the four pts whose position vectors are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$ are coplanar if and only if

$$[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = [\vec{\beta}\,\vec{\gamma}\,\vec{\delta}] + [\vec{\gamma}\,\vec{\alpha}\,\vec{\delta}] + [\vec{\alpha}\,\vec{\beta}\,\vec{\delta}]$$

Solution: Let A, B, C & D be four pts whose position vectors are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$, & $\vec{\delta}$ respectively.

then A, B, C, D are coplanar iff $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are coplanar

Now $\vec{AB} = \vec{\beta} - \vec{\alpha}$, $\vec{AC} = \vec{\gamma} - \vec{\alpha}$, $\vec{AD} = \vec{\delta} - \vec{\alpha}$

Then $\vec{AB}$, $\vec{AC}$ & $\vec{AD}$ will be coplanar iff $[\vec{AB}\,\vec{AC}\,\vec{AD}] = 0$

$$\text{ie iff } (\vec{\beta} - \vec{\alpha}) \cdot \{(\vec{\gamma} - \vec{\alpha}) \times (\vec{\delta} - \vec{\alpha})\} = 0$$

$$\text{ie iff } (\vec{\beta} - \vec{\alpha}) \cdot \{\vec{\gamma} \times \vec{\delta} - \vec{\alpha} \times \vec{\delta} - \vec{\gamma} \times \vec{\alpha}\} = 0$$

$$\text{ie iff } [\vec{\beta}\,\vec{\gamma}\,\vec{\delta}] - [\vec{\beta}\,\vec{\alpha}\,\vec{\delta}] - [\vec{\beta}\,\vec{\gamma}\,\vec{\alpha}] - [\vec{\alpha}\,\vec{\gamma}\,\vec{\delta}] = 0$$

$$\text{ie iff } [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = [\vec{\beta}\,\vec{\gamma}\,\vec{\delta}] + [\vec{\gamma}\,\vec{\alpha}\,\vec{\delta}] + [\vec{\alpha}\,\vec{\beta}\,\vec{\delta}]$$

Thus four pts whose position vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ & $\vec{\delta}$ are coplanar iff $[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = [\vec{\beta}\,\vec{\gamma}\,\vec{\delta}] + [\vec{\gamma}\,\vec{\alpha}\,\vec{\delta}] + [\vec{\alpha}\,\vec{\beta}\,\vec{\delta}]$

17(c) If the four pts vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ be such that $\vec{a}+\vec{b}+\vec{c}+\vec{d}=\vec{0}$, then show that $\dfrac{|\vec{a}|}{[\vec{\beta}\,\vec{\gamma}\,\vec{\delta}]} = \dfrac{-|\vec{b}|}{[\vec{\gamma}\,\vec{\delta}\,\vec{\alpha}]} = \dfrac{|\vec{c}|}{[\vec{\delta}\,\vec{\alpha}\,\vec{\beta}]} = \dfrac{-|\vec{d}|}{[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]}$ where $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, $\vec{\delta}$ are unit vectors along $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ respectively.

Solution: Since $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ & $\vec{\delta}$ are unit vectors along $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ respectively, so we can write

$$\vec{a} = |\vec{a}|\,\vec{\alpha}$$
$$\vec{b} = |\vec{b}|\,\vec{\beta}$$
$$\vec{c} = |\vec{c}|\,\vec{\gamma}$$
$$\& \ \vec{d} = |\vec{d}|\,\vec{\delta}$$

Since $\vec{a}+\vec{b}+\vec{c}+\vec{d}=\vec{0}$, so $|\vec{a}|\vec{\alpha}+|\vec{b}|\vec{\beta}+|\vec{c}|\vec{\gamma}+|\vec{d}|\vec{\delta} = \vec{0}$ ... (I).

taking dot product from both sides of (I) by $(\vec{\beta}\times\vec{\delta})$

$$|\vec{a}|\,[\vec{\alpha}\,\vec{\beta}\,\vec{\delta}] + |\vec{c}|\,[\vec{\gamma}\,\vec{\beta}\,\vec{\delta}] = 0$$

or $$\frac{|\vec{a}|}{[\vec{\beta}\,\vec{\gamma}\,\vec{\delta}]} = \frac{|\vec{c}|}{[\vec{\delta}\,\vec{\alpha}\,\vec{\beta}]} \quad \text{... (II)}$$

Again taking dot product from both sides of (I) by $(\vec{\alpha}\times\vec{\gamma})$ we get $|\vec{b}|\,[\vec{\beta}\,\vec{\alpha}\,\vec{\gamma}] + |\vec{d}|\,[\vec{\delta}\,\vec{\alpha}\,\vec{\gamma}] = 0$

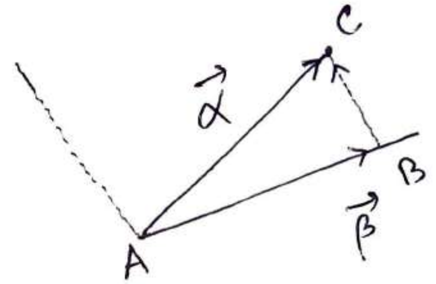
or $$\frac{-|\vec{b}|}{[\vec{\gamma}\,\vec{\delta}\,\vec{\alpha}]} = \frac{-|\vec{d}|}{[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]} \quad \text{... (III)}$$

Again taking dot product from both sides of (I) by $(\vec{\alpha}\times\vec{\delta})$ we get $|\vec{b}|\,[\vec{\beta}\,\vec{\alpha}\,\vec{\delta}] + |\vec{c}|\,[\vec{\gamma}\,\vec{\alpha}\,\vec{\delta}] = 0 \Rightarrow \dfrac{|\vec{c}|}{[\vec{\delta}\,\vec{\alpha}\,\vec{\beta}]} = \dfrac{-|\vec{b}|}{[\vec{\gamma}\,\vec{\delta}\,\vec{\alpha}]}$ (IV)

* From (II), (III) & (IV) we get the required result.

16(b). If a vector $\vec{\alpha}$ be resolved into components parallel and perpendicular to another vector $\vec{\beta}$, then show that the components are $\dfrac{\vec{\beta}\cdot\vec{\alpha}}{|\vec{\beta}|^2}\vec{\beta}$ and $\dfrac{\vec{\beta}\times(\vec{\alpha}\times\vec{\beta})}{|\vec{\beta}|^2}$ respectively

Solution :



Here components of $\vec{\alpha}\,(=\overrightarrow{AC})$ along $\vec{\beta}$ is $\overrightarrow{AB}$ and perpendicular to $\vec{\beta}$ is $\overrightarrow{BC}$

~~The projection of $\vec{\alpha}$ along~~

Let $\overrightarrow{AB} = c\vec{\beta}$. Then $c\vec{\beta} + \overrightarrow{BC} = \vec{\alpha}$ ...(1)

taking dot product from both sides of (1) by $\vec{\beta}$ we get $c\,\vec{\beta}\cdot\vec{\beta} + 0 = \vec{\alpha}\cdot\vec{\beta}$ [Since $\vec{\beta}$ and $\overrightarrow{BC}$ are orthogonal]

or $c = \dfrac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|^2}$

Hence components of $\vec{\alpha}$ along $\vec{\beta}$ is $c\vec{\beta} = \dfrac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|^2}\vec{\beta}$

Again, from (1) we get

$$\overrightarrow{BC} = \vec{\alpha} - c\vec{\beta} = \vec{\alpha} - \frac{\vec{\alpha}\cdot\vec{\beta}}{|\vec{\beta}|^2}\vec{\beta}$$

$$= \frac{|\vec{\beta}|^2\vec{\alpha} - (\vec{\alpha}\cdot\vec{\beta})\vec{\beta}}{|\vec{\beta}|^2} = \frac{\vec{\beta}\times(\vec{\alpha}\times\vec{\beta})}{|\vec{\beta}|}$$

Hence component of $\vec{\alpha}$ perpendicular to $\vec{\beta}$ is $\dfrac{\vec{\beta}\times(\vec{\alpha}\times\vec{\beta})}{|\vec{\beta}|}$.

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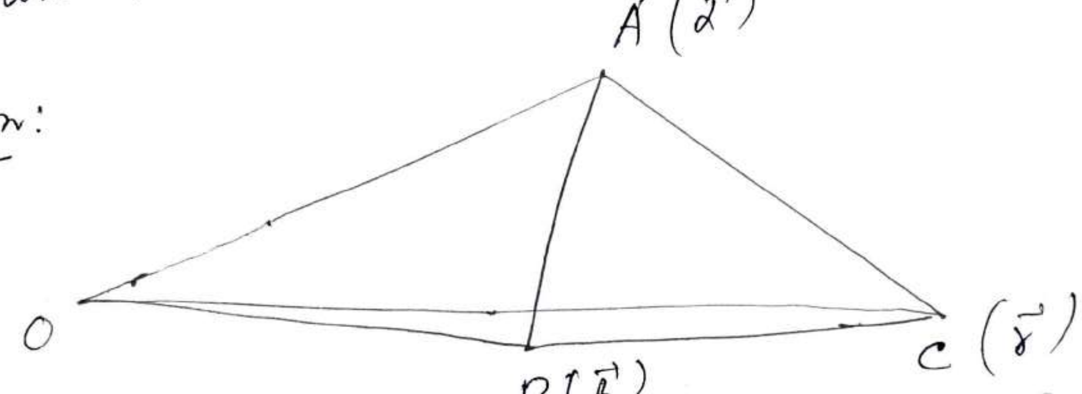
12(a) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be three vectors from the origin to the pts A, B, C respectively, then show that the vector $(\vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} + \vec{\alpha} \times \vec{\beta})$ is perpendicular to the plane ABC

Solution:



Here O is the base pt. Then p.v of A, B and C are respectively $(\vec{\alpha})$, $\vec{\beta}$ & $\vec{\gamma}$

Then
$$\overrightarrow{AB} = (\vec{\beta} - \vec{\alpha})$$
$$\overrightarrow{AC} = \vec{\gamma} - \vec{\alpha}$$
$$\overrightarrow{BC} = \vec{\gamma} - \vec{\beta}$$

Then $(\vec{\beta} - \vec{\alpha})$, $(\vec{\gamma} - \vec{\alpha})$ and $(\vec{\gamma} - \vec{\beta})$ lie on the plane ABC.

Then any vector $\vec{d}$ on the plane ABC can be put as

$$\vec{d} = c_1(\vec{\beta}-\vec{\alpha}) + c_2(\vec{\gamma}-\vec{\alpha}) + c_3(\vec{\gamma}-\vec{\beta})$$

Now, $(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta}) \cdot \left(c_1(\vec{\beta}-\vec{\alpha}) + c_2(\vec{\gamma}-\vec{\alpha}) + c_3(\vec{\gamma}-\vec{\beta})\right)$

$$= c_1(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})\cdot(\vec{\beta}-\vec{\alpha}) + c_2$$

$$(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}-\vec{\alpha}) +$$

$$c_3(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}-\vec{\beta})$$

$$= c_1\left([\vec{\beta}\,\vec{\gamma}\,\vec{\alpha}] - [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]\right) + c_2(\ \ )$$

$$+ c_3(\ \ )$$

$$= 0.$$

$\therefore$ $\vec{d}$ is perpendicular to the vector $(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})$.

16 (a) Prove that three mutually perpendicular unit vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ form, in the ~~form~~ given order, a right handed system iff $[\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = 1$

Solution: Let us assume that $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ form a right handed system.

Then we get $\vec{\alpha} \times \vec{\beta} = \vec{\gamma}$, $\vec{\beta} \times \vec{\gamma} = \vec{\alpha}$ and $\vec{\gamma} \times \vec{\alpha} = \vec{\beta}$

Now, $[\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = \vec{\alpha} \cdot (\vec{\beta} \times \vec{\gamma})$

$= \vec{\alpha} \cdot \vec{\alpha}$

$= |\vec{\alpha}|^2$

$= 1 \quad (\because |\vec{\alpha}| = 1)$

which is one part.

Conversely let $[\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = 1$. We shall show that $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ form a right handed system in the given order.

For this we need to show that $\vec{\alpha} \times \vec{\beta} = \vec{\gamma}$, $\vec{\beta} \times \vec{\gamma} = \vec{\alpha}$ & $\vec{\gamma} \times \vec{\alpha} = \vec{\beta}$.

Since $\vec{\alpha}$, $\vec{\beta}$ & $\vec{\gamma}$ are mutually perpendicular to each other, so $\vec{\beta} \times \vec{\gamma} = K \vec{\alpha}$

$$\Rightarrow \vec{\alpha} \cdot (\vec{\beta} \times \vec{\gamma}) = K \vec{\alpha} \cdot \vec{\alpha}$$

$$\Rightarrow [\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = K |\vec{\alpha}|^2$$

$$\Rightarrow K = [\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = 1 \quad \left(\because |\vec{\alpha}|^2 = 1 \text{ and } [\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = 1 \right)$$

$$\therefore \vec{\beta} \times \vec{\gamma} = \vec{\alpha}$$

Again we can write $\vec{\alpha} \times \vec{\beta} = \lambda \vec{\gamma}$

$$\Rightarrow (\vec{\alpha} \times \vec{\beta}) \cdot \vec{\gamma} = \lambda \quad [\because |\vec{\gamma}| = 1]$$

$$\Rightarrow \lambda = [\vec{\gamma}\, \vec{\alpha}\, \vec{\beta}] = [\vec{\alpha}\, \vec{\beta}\, \vec{\gamma}] = 1$$

$$\therefore \vec{\alpha} \times \vec{\beta} = \vec{\gamma}$$

Similarly we get $\vec{\gamma} \times \vec{\alpha} = \vec{\beta}$

Thus $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$, in the given order, form a right handed system.

18(iii) Shortcut, $[\vec{p}\,\vec{q}\,\vec{r}][\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} \vec{p}\cdot\vec{a} & \vec{p}\cdot\vec{b} & \vec{p}\cdot\vec{c} \\ \vec{q}\cdot\vec{a} & \vec{q}\cdot\vec{b} & \vec{q}\cdot\vec{c} \\ \vec{r}\cdot\vec{a} & \vec{r}\cdot\vec{b} & \vec{r}\cdot\vec{c} \end{vmatrix}$

Solution: Let $\vec{p} = p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$

$\vec{q} = q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

$\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Then $[\vec{p}\,\vec{q}\,\vec{r}] = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$

and $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Now, $[\vec{p}\,\vec{q}\,\vec{r}]\,[\vec{a}\,\vec{b}\,\vec{c}]$

$$= \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} p_1a_1+p_2a_2+p_3a_3 & p_1b_1+p_2b_2+p_3b_3 & p_1c_1+p_2c_2+p_3c_3 \\ q_1a_1+q_2a_2+q_3a_3 & q_1b_1+q_2b_2+q_3b_3 & q_1c_1+q_2c_2+q_3c_3 \\ r_1a_1+r_2a_2+r_3a_3 & r_1b_1+r_2b_2+r_3b_3 & r_1c_1+r_2c_2+r_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{p}\cdot\vec{a} & \vec{p}\cdot\vec{b} & \vec{p}\cdot\vec{c} \\ \vec{q}\cdot\vec{a} & \vec{q}\cdot\vec{b} & \vec{q}\cdot\vec{c} \\ \vec{r}\cdot\vec{a} & \vec{r}\cdot\vec{b} & \vec{r}\cdot\vec{c} \end{vmatrix}$$ (proved).

11 (b) Reduce the expression $(\vec{\beta}+\vec{\gamma})\cdot\{(\vec{\gamma}+\vec{\alpha})\times(\vec{\alpha}+\vec{\beta})\}$ to its simplest form and prove that it vanishes when $\vec{\alpha}$, $\vec{\beta}$ & $\vec{\gamma}$ are coplanar

Solution: $(\vec{\beta}+\vec{\gamma})\cdot\{(\vec{\gamma}+\vec{\alpha})\times(\vec{\alpha}+\vec{\beta})\}$

$$= (\vec{\beta}+\vec{\gamma})\cdot\{\vec{\gamma}\times\vec{\alpha}+\vec{\gamma}\times\vec{\beta}+\vec{0}+\vec{\alpha}\times\vec{\beta}\}$$

$$= [\vec{\beta}\,\vec{\gamma}\,\vec{\alpha}] + [\vec{\beta}\,\vec{\gamma}\,\vec{\beta}] + [\vec{\beta}\,\vec{\alpha}\,\vec{\beta}] + [\vec{\gamma}\,\vec{\gamma}\,\vec{\alpha}] + [\vec{\gamma}\,\vec{\gamma}\,\vec{\beta}] + [\vec{\gamma}\,\vec{\alpha}\,\vec{\beta}]$$

$$= 2[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] \longrightarrow (1)$$

which is the simplest form of ~~(a)~~ the given expression.

2nd part: Since $\vec{\alpha}$, $\vec{\beta}$ & $\vec{\gamma}$ are coplanar, so $[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$. Then from (1), we get

$$(\vec{\beta}+\vec{\gamma})\cdot\{(\vec{\gamma}+\vec{\alpha})\times(\vec{\alpha}+\vec{\beta})\} = 0.$$

which is the required result.

11(c) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$, then find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, $\vec{b}$ & $\vec{c}$ being non parallel.

Solution: $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{1}{2}\vec{b}$

or $(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = \frac{1}{2}\vec{b}$ ...(1)

Then taking cross product from both sides of (1) by $\vec{c}$ we get

$$(\vec{a}\cdot\vec{c})(\vec{b}\times\vec{c}) - (\vec{a}\cdot\vec{b})\vec{c}\times\vec{c} = \frac{1}{2}\vec{b}\times\vec{c}$$

or $(\vec{a}\cdot\vec{c} - \frac{1}{2})(\vec{b}\times\vec{c}) = \vec{0}$

$\therefore$ $\vec{a}\cdot\vec{c} - \frac{1}{2} = 0$ (since $\vec{b}\times\vec{c} \neq \vec{0}$ as $\vec{b}$ & $\vec{c}$ are non parallel unit vector).

or $|\vec{a}||\vec{c}|\cos(\vec{a},\vec{c}) = \frac{1}{2}$

or $(\vec{a},\vec{c}) = \frac{\pi}{3}$. where $(\vec{a},\vec{c})$ denote the angle between the vectors $\vec{a}$ & $\vec{c}$.

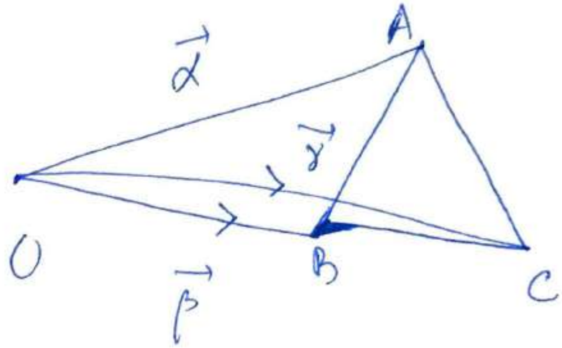
Again taking cross product from both sides of (1) by $\vec{b}$ we get $(\vec{a}\cdot\vec{b})(\vec{c}\times\vec{b}) = \vec{0}$ $\Rightarrow$ $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos(\vec{a},\vec{b}) = 0$

$\therefore (\vec{a},\vec{b}) = \frac{\pi}{2}$

12(a) If $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ be three vectors from the origin to the pts A, B & C respectively. Then show that the vectors $(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})$ is perpendicular to the plane ABC.

triple pro



Solution

Here, $\overrightarrow{AB}$ = p.v of the pt B − p.v of the pt A

$$= \vec{\beta} - \vec{\alpha}$$

Then $\overrightarrow{AB}\cdot(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})$

$$= (\vec{\beta}-\vec{\alpha})\cdot(\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta})$$

$$= [\vec{\beta}\,\vec{\beta}\,\vec{\gamma}] + [\vec{\beta}\,\vec{\gamma}\,\vec{\alpha}] + [\vec{\beta}\,\vec{\alpha}\,\vec{\beta}] - [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] + [\vec{\alpha}\,\vec{\gamma}\,\vec{\alpha}] - [\vec{\alpha}\,\vec{\alpha}\,\vec{\beta}]$$

$$= [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] - [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$$

$\therefore\ \vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\alpha} + \vec{\alpha}\times\vec{\beta}$ is perpendicular to $\overrightarrow{AB}$.

Similarly ~~Bc~~ $\vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} + \vec{\alpha} \times \vec{\beta}$ is perpendicular to both $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

Since $\vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} + \vec{\alpha} \times \vec{\beta}$ is perpendicular to $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ vectors, so $\vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} + \vec{\alpha} \times \vec{\beta}$ is perpendicular to the plane of ABC.

Show that the vectors $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.

Solution: Let $\vec{\alpha} = \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\vec{\beta} = \vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$

$\vec{\gamma} = \vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

Then $[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = \vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma})$

$$= \vec{\alpha}\cdot\left\{\left[(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}\right]\times\left[(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}\right]\right\}$$

$$= \vec{\alpha}\cdot\left\{(\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{b})(\vec{c}\times\vec{a}) - (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{a})(\vec{c}\times\vec{b}) + (\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})(\vec{a}\times\vec{b})\right\}$$

$$= (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{b})\,\vec{\alpha}\cdot(\vec{c}\times\vec{a}) - (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{a})(\vec{\alpha}\cdot\vec{c}\times\vec{b}) + (\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})(\vec{\alpha}\cdot(\vec{a}\times\vec{b}))$$

$$= (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{b})\left\{\left((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\right)\cdot(\vec{c}\times\vec{a})\right\}$$
$$- (\vec{b}\cdot\vec{a})(\vec{c}\cdot\vec{a})\left\{\left[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\right]\cdot(\vec{c}\times\vec{b})\right\}$$
$$+ (\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})\left\{\left[(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}\right]\cdot(\vec{a}\times\vec{b})\right\}$$

$$= (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})[\vec{b}\,\vec{c}\,\vec{a}] - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})[\vec{c}\,\vec{a}\,\vec{b}]$$

$= 0 \quad \left[\because [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]\right]$, so $\vec{\alpha}, \vec{\beta}$ & $\vec{\gamma}$ are coplanar.

13(c) Prove that any vector $\vec{d}$ can be expressed as

$$\vec{d} = \frac{(\vec{a}\cdot\vec{d})(\vec{b}\times\vec{c}) + (\vec{b}\cdot\vec{d})(\vec{c}\times\vec{a}) + (\vec{c}\cdot\vec{d})(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

where $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar vectors.

Solution: Since $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar, so $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ are also non coplanar. Then any vector $\vec{d}$ can be expressed as the linear combination of the vectors $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$.

Let $\vec{d} = \lambda_1(\vec{a}\times\vec{b}) + \lambda_2(\vec{b}\times\vec{c}) + \lambda_3(\vec{c}\times\vec{a})$ ... (1)

where $\lambda_1, \lambda_2, \lambda_3$ are three scalars to be determined.

taking dot product from both sides of (1) by $\vec{a}$, we get

$$\vec{a}\cdot\vec{d} = \lambda_2[\vec{a}\,\vec{b}\,\vec{c}]$$

$$\therefore \lambda_2 = \frac{\vec{a}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Again taking dot product from both sides of (1) by $\vec{b}$ we get

$$\vec{b}\cdot\vec{d} = \lambda_3[\vec{b}\,\vec{c}\,\vec{a}]$$

$$\therefore \lambda_3 = \frac{\vec{b}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly $\lambda_1 = \dfrac{\vec{c}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}$.

Then putting the values of $\lambda_1, \lambda_2, \lambda_3$ in (1) we get

$$\vec{d} = \frac{\vec{c}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}\times\vec{b}) + \frac{\vec{a}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{b}\cdot\vec{d}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{c}\times\vec{a})$$

$$= \frac{(\vec{c}\cdot\vec{d})(\vec{a}\times\vec{b}) + (\vec{a}\cdot\vec{d})(\vec{b}\times\vec{c}) + (\vec{b}\cdot\vec{d})(\vec{c}\times\vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

13(a) If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non coplanar vectors, then show that $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ & $\vec{a}\times\vec{b}$ are also non coplanar.

Solution: Since $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar, so

$[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

Now $[(\vec{b}\times\vec{c})(\vec{c}\times\vec{a})(\vec{a}\times\vec{b})]$

$$= (\vec{b}\times\vec{c})\cdot\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}$$

$$= (\vec{b}\times\vec{c})\cdot\{((\vec{c}\times\vec{a})\cdot\vec{b})\vec{a} - ((\vec{c}\times\vec{a})\cdot\vec{a})\vec{b}\}$$

$$= (\vec{b}\times\vec{c})\cdot\{[\vec{b}\,\vec{c}\,\vec{a}]\,\vec{a} - \vec{0}\} \quad [\because [\vec{a}\,\vec{c}\,\vec{a}] = 0]$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]((\vec{b}\times\vec{c})\cdot\vec{a})$$

$$= [\vec{b}\,\vec{c}\,\vec{a}][\vec{a}\,\vec{b}\,\vec{c}]$$

$$= [\vec{a}\,\vec{b}\,\vec{c}][\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 \neq 0$$

$$(\because [\vec{a}\,\vec{b}\,\vec{c}] \neq 0).$$

This shows that $(\vec{b}\times\vec{c})$, $(\vec{c}\times\vec{a})$ and $(\vec{a}\times\vec{b})$ are non coplanar.

13(b) Show that, three non coplanar vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$ can be put as $\vec{a} = \dfrac{\vec{a}\cdot\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{b}\times\vec{c}) + \dfrac{\vec{a}\cdot b}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{c}\times\vec{a})$

$$+ \frac{\vec{a}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}\times\vec{b})$$

Solution: Since $\vec{a}, \vec{b}$ & $\vec{c}$ are non coplanar, so $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ & $\vec{c}\times\vec{a}$ are also non coplanar (see 13(a)). Since any vector can be expressed as the linear combination of the ~~vect~~ three non coplanar vectors, so we assume

$$\vec{a} = l(\vec{a}\times\vec{b}) + m(\vec{b}\times\vec{c}) + n(\vec{c}\times\vec{a}) \quad \text{--(1)}$$

where l, m & n are suitable constants.

Taking dot product by $\vec{a}$ from both sides of (1), we get

$$\vec{a}.\vec{a} = l[\vec{a}\,\vec{a}\,\vec{b}] + m[\vec{a}\,\vec{b}\,\vec{c}] + n[\vec{a}\,\vec{c}\,\vec{a}]$$

or $$\vec{a}.\vec{a} = m[\vec{a}\,\vec{b}\,\vec{c}] \quad \left[\because [\vec{a}\,\vec{a}\,\vec{b}] = [\vec{a}\,\vec{c}\,\vec{a}] = 0\right]$$

$$\therefore m = \frac{\vec{a}.\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Again taking dot product by $\vec{b}$ from both sides of (1) we get $\vec{a}.\vec{b} = n[\vec{b}\,\vec{c}\,\vec{a}]$

$$\therefore n = \frac{\vec{a}.\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly $\vec{a}.\vec{c} = l[\vec{c}\,\vec{a}\,\vec{b}]$

$$\therefore l = \frac{\vec{a}.\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Thus from (1) we have

$$\vec{a} = \frac{\vec{a}.\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{b}\times\vec{c}) + \frac{\vec{a}.\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{c}\times\vec{a}) + \frac{\vec{a}.\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}\times\vec{b})$$

(viii) Page 70 Show that $[\vec{l}\ \vec{m}\ \vec{n}](\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$ where $\vec{l}$, $\vec{m}$ & $\vec{n}$ are three non coplanar vectors.

Solution: Since $\vec{l}$, $\vec{m}$ & $\vec{n}$ are three non coplanar vectors

so, $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$ & $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$

where $a_i\ (i=1,2,3)$ & $b_i\ (i=1,2,3)$ are scalars.

Now $\vec{a}\times\vec{b} = (a_1\vec{l} + a_2\vec{m} + a_3\vec{n}) \times (b_1\vec{l} + b_2\vec{m} + b_3\vec{n})$

$= a_1b_2\,\vec{l}\times\vec{m} + a_1b_3\,\vec{l}\times\vec{n} + a_2b_1\,\vec{m}\times\vec{l} + a_2b_3\,\vec{m}\times\vec{n} + a_3b_1\,\vec{n}\times\vec{l} + a_3b_2\,\vec{n}\times\vec{m}$

$= (a_1b_2 - b_1a_2)\,\vec{l}\times\vec{m} + (a_2b_3 - b_2a_3)\,\vec{m}\times\vec{n} + (a_3b_1 - b_3a_1)\,\vec{n}\times\vec{l}$

Again $\begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix}$

$= (\vec{l}\cdot\vec{a})\{(\vec{m}\cdot\vec{b})\vec{n} - (\vec{n}\cdot\vec{b})\vec{m}\} - (\vec{m}\cdot\vec{a})\{(\vec{l}\cdot\vec{b})\vec{n} - (\vec{n}\cdot\vec{b})\vec{l}\} + (\vec{n}\cdot\vec{a})\{(\vec{l}\cdot\vec{b})\vec{m} - (\vec{m}\cdot\vec{b})\vec{l}\}$

$= (\vec{l}\cdot\vec{a})\{\vec{b}\times(\vec{n}\times\vec{m})\} - (\vec{m}\cdot\vec{a})\{\vec{b}\times(\vec{n}\times\vec{l})\}$
$+ (\vec{n}\cdot\vec{a})\{\vec{b}\times(\vec{m}\times\vec{l})\}$

$$= \vec{b} \times \{ (\vec{l}\cdot\vec{a})(\vec{n}\times\vec{m}) - (\vec{m}\cdot\vec{a})(\vec{n}\times\vec{l}) + (\vec{n}\cdot\vec{a})(\vec{m}\times\vec{l}) \}$$

$$= -\vec{b} \times \{ (\vec{n}\cdot\vec{a})(\vec{l}\times\vec{m}) + (\vec{l}\cdot\vec{a})(\vec{m}\times\vec{n}) + (\vec{m}\cdot\vec{a})(\vec{n}\times\vec{l}) \} \quad \ldots (1)$$

Since $\vec{l}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors, so $\vec{l}\times\vec{m}$, $\vec{m}\times\vec{n}$, $\vec{n}\times\vec{l}$ are also non coplanar, then we can write

$$\vec{a} = c_1(\vec{l}\times\vec{m}) + c_2(\vec{m}\times\vec{n}) + c_3(\vec{n}\times\vec{l}) \quad \text{— (11)}$$

where c_1 & c_2 & c_3 are scalar to be determined.

Taking dot product from both sides of (11) by $\vec{l}$ we get

$$\vec{l}\cdot\vec{a} = c_2 [\vec{l}\,\vec{m}\,\vec{n}]$$

$$\therefore c_2 = \frac{\vec{l}\cdot\vec{a}}{[\vec{l}\,\vec{m}\,\vec{n}]} \quad (\because [\vec{l}\,\vec{m}\,\vec{n}] \neq 0)$$

Similarly $c_3 = \dfrac{\vec{m}\cdot\vec{a}}{[\vec{l}\,\vec{m}\,\vec{n}]}$ and $c_1 = \dfrac{\vec{n}\cdot\vec{a}}{[\vec{l}\,\vec{m}\,\vec{n}]}$.

Then from (11) we get

$$\vec{a} = \frac{(\vec{n}\cdot\vec{a})(\vec{l}\times\vec{m}) + (\vec{l}\cdot\vec{a})(\vec{m}\times\vec{n}) + (\vec{m}\cdot\vec{a})(\vec{n}\times\vec{l})}{[\vec{l}\,\vec{m}\,\vec{n}]}$$

$$\therefore [\vec{l}\,\vec{m}\,\vec{n}]\,\vec{a} = (\vec{n}\cdot\vec{a})(\vec{l}\times\vec{m}) + (\vec{l}\cdot\vec{a})(\vec{m}\times\vec{n}) + (\vec{m}\cdot\vec{a})(\vec{n}\times\vec{l})$$

Then from

$$\begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n} \end{vmatrix} = -\vec{b} \times \left([\vec{l}\,\vec{m}\,\vec{n}]\,\vec{a} \right)$$

$$= [\vec{l}\,\vec{m}\,\vec{n}](\vec{a}\times\vec{b}).$$

14 (b) If $\vec{u}, \vec{v}, \vec{w}$ be three mutually perpendicular unit vectors s.t $\vec{u} \times \vec{v} = \vec{w}$, then prove that $\vec{v} = \vec{w} \times \vec{u}$ and $\vec{u} = \vec{v} \times \vec{w}$

Solution: Since $\vec{w} = \vec{u} \times \vec{v}$ - - - (1)

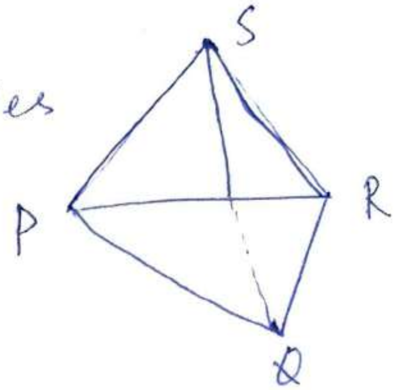
Taking cross product from both sides of (1) by $\vec{u}$

we get
$$\vec{w} \times \vec{u} = (\vec{u} \times \vec{v}) \times \vec{u}$$
$$\text{or} \quad \vec{w} \times \vec{u} = |\vec{u}|^2 \vec{v} - (\vec{v} \cdot \vec{u})\, \vec{u}$$
$$\text{or} \quad \vec{w} \times \vec{u} = \vec{v} \quad [\because |\vec{u}| = 1 \text{ and } \vec{u} \cdot \vec{v} = 0]$$

Again taking cross product from both sides of (1) by $\vec{v}$

we get
$$\vec{v} \times \vec{w} = \vec{v} \times (\vec{u} \times \vec{v})$$
$$\text{or} \quad \vec{v} \times \vec{w} = |\vec{v}|^2 \vec{u} - (\vec{v} \cdot \vec{u})\, \vec{v}$$
$$\text{or} \quad \vec{v} \times \vec{w} = \vec{u} \quad [\because |\vec{v}| = 1 \text{ and } \vec{v} \cdot \vec{u} = 0]$$

10(a) $P(1,3,-1)$, $Q(0,1,6)$, $R(-1,3,1)$ are three pts in space. Find the co-ordinates of a pt S on the y-axis, such that the volume of the tetrahedron $PQRS$ is 10 cubic units



Solution: Since the pt S lies on the y-axis, so we take co-ordinates of S as $(0, y, 0)$.

Then $\overrightarrow{PS}$ = p.v of S − p.v of P

$$= (0, y, 0) - (1, 3, -1)$$

$$= (-1, y-3, 1)$$

$\overrightarrow{PR}$ = p.v of R − p.v of P

$$= (-1, 3, 1) - (1, 3, -1)$$

$$= (-2, 0, 2)$$

& $\overrightarrow{PQ}$ = p.v of Q − p.v of P

$$= (0, 1, 6) - (1, 3, -1)$$

$$= (-1, -2, 7)$$

Then volume of the tetrahedron $PQRS$ is

$$= \frac{1}{6}\left[\overrightarrow{PS}\ \overrightarrow{PR}\ \overrightarrow{PQ}\right]$$

$$= \frac{1}{6}\begin{vmatrix} -1 & y-3 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 7 \end{vmatrix}$$

$$= \frac{1}{6}\left\{ -1\left(0+4\right) - (y-3)\left(+2-14\right) + 1\,(4) \right\}$$

$$= \frac{1}{6}\left\{ -4 + 12(y-3) + 4 \right\}$$

$$= \frac{1}{6}\left\{ 12y - 36 \right\} = 2y - 6$$

According to the problem,

$$2y - 6 = \pm 10$$

$$\text{or } \; 2y = \pm 10 + 6$$

$$\therefore y = 8, -2$$

20 (vi) Show that $|\vec{\alpha}\times\vec{\beta}|^2 |\vec{\alpha}\times\vec{\gamma}|^2 - \{(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})\}$

$$= |\vec{\alpha}|^2 \, [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]^2$$

Solution:

$$|\vec{\alpha}\times\vec{\beta}|^2$$

$$= (\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\beta})$$

$$= \{(\vec{\alpha}\times\vec{\beta})\times\vec{\alpha}\}\cdot\vec{\beta}$$ (by dot and cross interchanging property).

$$= \{(\vec{\alpha}\cdot\vec{\alpha})\vec{\beta} - (\vec{\beta}\cdot\vec{\alpha})\vec{\alpha}\}\cdot\vec{\beta}$$

$$= (\vec{\alpha}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\beta}) - (\vec{\beta}\cdot\vec{\alpha})(\vec{\alpha}\cdot\vec{\beta})$$

$$= |\vec{\alpha}|^2 |\vec{\beta}|^2 - \{(\vec{\alpha}\cdot\vec{\beta})\}^2$$

Similarly $|\vec{\alpha}\times\vec{\gamma}|^2 = |\vec{\alpha}|^2 |\vec{\gamma}|^2 - \{(\vec{\alpha}\cdot\vec{\gamma})\}^2$

Now, $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})$

$$= \{(\vec{\alpha}\times\vec{\beta})\times\vec{\alpha}\}\cdot\vec{\gamma}$$

$$= \{(\vec{\alpha}\cdot\vec{\alpha})\vec{\beta} - (\vec{\beta}\cdot\vec{\alpha})\vec{\alpha}\}\cdot\vec{\gamma}$$

$$= |\vec{\alpha}|^2(\vec{\beta}\cdot\vec{\gamma}) - (\vec{\beta}\cdot\vec{\alpha})(\vec{\alpha}\cdot\vec{\gamma})$$

Now, $L.H.S = |\vec{\alpha}\times\vec{\beta}|^2|\vec{\alpha}\times\vec{\gamma}|^2 - \{(\vec{\alpha}\times\vec{\beta})(\vec{\alpha}\times\vec{\gamma})\}^2$

$$= \{(\vec{\alpha}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\beta}) - \{(\vec{\alpha}\cdot\vec{\beta})\}^2\}\{(\vec{\alpha}\cdot\vec{\alpha})(\vec{\gamma}\cdot\vec{\gamma}) - \{(\vec{\alpha}\cdot\vec{\gamma})\}^2\}$$

$$- \{(\vec{\alpha}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\gamma}) - (\vec{\beta}\cdot\vec{\alpha})(\vec{\alpha}\cdot\vec{\gamma})\}^2$$

$$= \{(\vec{\alpha}\cdot\vec{\alpha})\}^2\left((\vec{\beta}\cdot\vec{\beta})(\vec{\gamma}\cdot\vec{\gamma}) - \{(\vec{\beta}\cdot\vec{\gamma})\}^2\right)$$

$$- \{(\vec{\alpha}\cdot\vec{\beta})\}^2(\vec{\alpha}\cdot\vec{\alpha})(\vec{\gamma}\cdot\vec{\gamma})$$

$$- (\vec{\alpha}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\beta})\{(\vec{\alpha}\cdot\vec{\gamma})\}^2$$

$$+ 2(\vec{\alpha}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\gamma})(\vec{\beta}\cdot\vec{\alpha})(\vec{\alpha}\cdot\vec{\gamma})$$

$$R.H.S = |\vec{\alpha}|^2[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]^2$$

$$= (\vec{\alpha}\cdot\vec{\alpha})\begin{vmatrix}\vec{\alpha}\cdot\vec{\alpha} & \vec{\alpha}\cdot\vec{\beta} & \vec{\alpha}\cdot\vec{\gamma}\\ \vec{\beta}\cdot\vec{\alpha} & \vec{\beta}\cdot\vec{\beta} & \vec{\beta}\cdot\vec{\gamma}\\ \vec{\gamma}\cdot\vec{\alpha} & \vec{\gamma}\cdot\vec{\beta} & \vec{\gamma}\cdot\vec{\gamma}\end{vmatrix}$$

$$= (\vec{\alpha}\cdot\vec{\alpha})\Big[(\vec{\alpha}\cdot\vec{\alpha})\{(\vec{\beta}\cdot\vec{\beta})(\vec{\gamma}\cdot\vec{\gamma}) - \{(\vec{\beta}\cdot\vec{\gamma})\}^2\} -$$

$$- (\vec{\alpha}\cdot\vec{\beta})\{(\vec{\beta}\cdot\vec{\alpha})(\vec{\gamma}\cdot\vec{\gamma}) - (\vec{\beta}\cdot\vec{\gamma})(\vec{\gamma}\cdot\vec{\alpha})\}$$

$$+ (\vec{\alpha}\cdot\vec{\gamma})\{(\vec{\beta}\cdot\vec{\alpha})(\vec{\beta}\cdot\vec{\gamma}) - (\vec{\beta}\cdot\vec{\beta})(\vec{\gamma}\cdot\vec{\alpha})$$

$$= \{(\vec{\alpha}\cdot\vec{\alpha})\}^2 \{ (\vec{\beta}\cdot\vec{\beta})(\vec{\gamma}\cdot\vec{\gamma}) - \{(\vec{\beta}\cdot\vec{\gamma})\}^2 \}$$

$$- \{(\vec{\alpha}\cdot\vec{\beta})\}^2 (\vec{\alpha}\cdot\vec{\alpha})(\vec{\gamma}\cdot\vec{\gamma}) +$$

$$2(\vec{\alpha}\cdot\vec{\alpha})(\vec{\alpha}\cdot\vec{\beta})(\vec{\beta}\cdot\vec{\gamma})(\vec{\gamma}\cdot\vec{\alpha}) - (\vec{\beta}\cdot\vec{\beta})\{(\vec{\gamma}\cdot\vec{\alpha})\}^2 (\vec{\alpha}\cdot\vec{\alpha})$$

$$= \text{L.H.S.}$$

So, $$|\vec{\alpha}\times\vec{\beta}|^2 |\vec{\alpha}\times\vec{\gamma}|^2 - \left((\vec{\alpha}\times\vec{\beta})\cdot(\vec{\alpha}\times\vec{\gamma})\right)^2$$

$$= |\vec{\alpha}|^2 [\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]^2 \text{ (proved)}.$$

Scalar tripil product

Find two vectors $\vec{a}$ & $\vec{b}$ which are of equal magnitude and mutually perpendicular, each have the x-component 5 and are each perpendicular to $3\hat{j}+4\hat{k}$

Solution: According to problem, we take

$$\vec{a} = 5\hat{i} + y_1\hat{j} + z_1\hat{k}$$

& $$\vec{b} = 5\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Since $\vec{a}$ & $\vec{b}$ are mutually perpendicular, so we get

$$\vec{a}\cdot\vec{b} = 0 \quad \text{i.e.} \quad 25 + y_1y_2 + z_1z_2 = 0 \cdots (1)$$

Again as $\vec{a}$ & $\vec{b}$ have equal magnitude, so we have

$$25 + y_1^2 + z_1^2 = 25 + y_2^2 + z_2^2$$

$$\Rightarrow y_1^2 + z_1^2 = y_2^2 + z_2^2 \cdots (2)$$

As $\vec{a}$ & $\vec{b}$ also perpendicular to the vector $(3\hat{j}+4\hat{k})$, Then

$$3y_1 + 4z_1 = 0 \cdots (3)$$

& $$3y_2 + 4z_2 = 0 \cdots (4)$$

with help of (3) & (4), (2) $\Rightarrow \frac{16}{9}z_1^2 + z_1^2 = \frac{16}{9}z_2^2 + z_2^2$

$$\Rightarrow z_1^2 = z_2^2$$

$$\Rightarrow z_1 = \pm z_2$$

When $z_1 = z_2$, Then from (1) we get

$$25 + \left(-\frac{4}{3}z_2\right)\left(-\frac{4}{3}z_2\right) + z_2^2 = 0 \quad \text{(with help of (3) \& (4))}$$

$$\Rightarrow 25 + \frac{16}{9}z_2^2 + z_2^2 = 0$$

$\Rightarrow z_2^2 = -9$, which gives imaginary values of z_2

Thus $z_1 = z_2$ is not possible.

Now, when $z_1 = -z_2$, from (1) we get

$$25 + \left(\frac{4}{3}z_1\right)\left(-\frac{4}{3}z_1\right) - z_1^2 = 0$$

$$\Rightarrow z_1^2 = 9 \Rightarrow z_1 = \pm 3$$

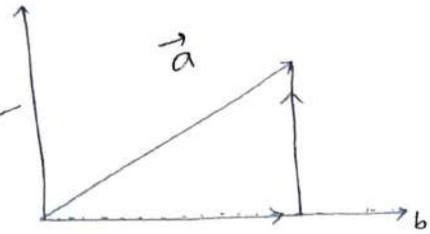
When $z_2 = 3$, from (4), $y_2 = -4$

and $z_1 = -3$, $y_1 = 4$ Hence two vector are

$$\vec{a} = 5\hat{i} + 4\hat{j} - 3\hat{k} \quad \& \quad \vec{b} = 5\hat{i} - 4\hat{j} + 3\hat{k}.$$

Express a vector $\vec{a}$ as the sum of two component vectors one parallel and the other perpendicular to the vector $\vec{b}$ in the form: $\vec{a} = \frac{1}{|\vec{b}|^2}\left((\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})\right)$

Solution:



Any vector perpendicular to the vector $\vec{b}$ and coplanar with $\vec{a}$ & $\vec{b}$ can be taken as

$$(\vec{b}\times\vec{a})\times\vec{b}$$
$$= (\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}$$

Now the component of $\vec{a}$ along a vector parallel to $\vec{b}$ is $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ Thus the vector parallel to $\vec{b}$ is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\left(\frac{\vec{b}}{|\vec{b}|}\right)$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}.$$

Again the component of the vector $\vec{a}$ ~~along~~ parallel to $(\vec{b}\times\vec{a})\times\vec{b}$

$$= \frac{\vec{a}\cdot\left((\vec{b}\times\vec{a})\times\vec{b}\right)}{|(\vec{b}\times\vec{a})\times\vec{b}|}$$

$$= \frac{(\vec{b}\cdot\vec{b})(\vec{a}\cdot\vec{a}) - (\vec{b}\cdot\vec{a})(\vec{b}\cdot\vec{a})}{|(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}|}$$

Thus the component vector of $\vec{a}$ parallel to $(\vec{b}\times\vec{a})\times\vec{b}$

$$\text{is} = \frac{(\vec{b}\cdot\vec{b})(\vec{a}\cdot\vec{a}) - (\vec{b}\cdot\vec{a})(\vec{b}\cdot\vec{a})}{\left\{|(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}|\right\}} \quad \frac{(\vec{b}\times\vec{a})\times\vec{b}}{|(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}|}$$

$$= \frac{(\vec{b}\cdot\vec{b})(\vec{a}\cdot\vec{a}) - (\vec{b}\cdot\vec{a})^2}{\left\{(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}\right\}\cdot\left\{(\vec{b}\cdot\vec{b})\vec{a} - (\vec{b}\cdot\vec{a})\vec{b}\right\}}\left((\vec{b}\times\vec{a})\times\vec{b}\right)$$

$$= \frac{\left(|\vec{b}|^2|\vec{a}|^2 - (\vec{b}\cdot\vec{a})^2\right)\left((\vec{b}\times\vec{a})\times\vec{b}\right)}{|\vec{b}|^2|\vec{b}|^2|\vec{a}|^2 - (\vec{b}\cdot\vec{b})(\vec{b}\cdot\vec{a})^2 - (\vec{b}\cdot\vec{a})^2(\vec{b}\cdot\vec{b}) + (\vec{b}\cdot\vec{a})^2(\vec{b}\cdot\vec{b})}$$

$$= \frac{\left(|\vec{b}|^2|\vec{a}|^2 - (\vec{b}\cdot\vec{a})^2\right)\left((\vec{b}\times\vec{a})\times\vec{b}\right)}{|\vec{b}|^2\left\{|\vec{b}|^2|\vec{a}|^2 - (\vec{b}\cdot\vec{a})^2\right\}} = \frac{1}{|\vec{b}|^2}\left((\vec{b}\times\vec{a})\times\vec{b}\right)$$

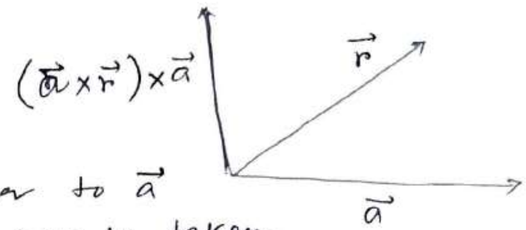
Thus $\vec{a}$ = component vector parallel to $\vec{b}$ + component vector parallel to $(\vec{b}\times\vec{a})\times\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} + \frac{1}{|\vec{b}|^2}\left((\vec{b}\times\vec{a})\times\vec{b}\right) = \frac{1}{|\vec{b}|^2}\left\{(\vec{a}\cdot\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})\right\}$$

Scalar tripil product (III (c))

4(2) Decompose a vector $\vec{r}$ as a linear combination of a vector $\vec{a}$ and another vector perpendicular to $\vec{a}$ and coplanar with $\vec{r}$ & $\vec{a}$.

Solution :



~~Solution~~

Any vector perpendicular to $\vec{a}$ and coplanar with $\vec{r}$ & $\vec{a}$ can be taken as $(\vec{a}\times\vec{r})\times\vec{a}$.

Now, we express $\vec{r}$ as the linear combination of vectors $\vec{a}$ & $(\vec{a}\times\vec{r})\times\vec{a}$.

Since $\vec{r}, \vec{a}$ & $(\vec{a}\times\vec{r})\times\vec{a}$ are coplanar, Hence there exist two scalar x & y such that

$$\vec{r} = x\vec{a} + y\,(\vec{a}\times\vec{r})\times\vec{a} \text{ ---- (ans)}$$

~~taking a~~ $$= x\vec{a} + y\left\{(\vec{a}\cdot\vec{a})\,\vec{r} - (\vec{a}\cdot\vec{r})\,\vec{a}\right\} \text{ ---- (1)}$$

taking dot product by $\vec{a}$ from both sides of (1) we get

$$\vec{r}\cdot\vec{a} = x(\vec{a}\cdot\vec{a}) + y\left\{(\vec{a}\cdot\vec{a})(\vec{r}\cdot\vec{a}) - (\vec{a}\cdot\vec{r})(\vec{a}\cdot\vec{a})\right\}$$

$$\Rightarrow |\vec{a}|^2 x = \vec{r}\cdot\vec{a}$$

$$\Rightarrow x = \frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}$$

Again taking cross product by $\vec{a}$ from both sides of (1) we get

$$\vec{r}\times\vec{a} = \vec{x}(\vec{a}\times\vec{a}) + y\left\{(\vec{a}\cdot\vec{a})(\vec{r}\times\vec{a}) - (\vec{a}\cdot\vec{r})(\vec{a}\times\vec{a})\right\}$$

$$\Rightarrow \vec{r}\times\vec{a} = y|\vec{a}|^2(\vec{r}\times\vec{a})$$

$$\Rightarrow (\vec{r}\times\vec{a})\left\{1 - |\vec{a}|^2 y\right\} = \vec{0}$$

$$\Rightarrow 1 - |\vec{a}|^2 y = 0 \quad [\because \vec{r}\times\vec{a} \neq \vec{0}]$$

$$\Rightarrow y = \frac{1}{|\vec{a}|^2}$$

Thus $$\vec{r} = \frac{\vec{r}\cdot\vec{a}}{|\vec{a}|^2}\,\vec{a} + \frac{1}{|\vec{a}|^2}\left((\vec{a}\times\vec{r})\times\vec{a}\right).$$

Scalar & vector product :

Find a unit vector parallel to the xy-plane & perpendicular to the vector $4\hat{i} - 3\hat{j} + \hat{k}$

Solution Any vector parallel to xy plane can be taken as

$\vec{r} = x\hat{i} + y\hat{j}$ ..(1) where x & y are some scalar and $\hat{i}$ & $\hat{j}$ are two unit vector along positive direction of x axis & y axis respectively.

Thus the unit vector along $\vec{r} = \dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}}$ . . . (2)

Let (2) be the required vector which is perpendicular to $4\hat{i} - 3\hat{j} + \hat{k}$

Then we get $\dfrac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} \cdot (4\hat{i} - 3\hat{j} + \hat{k}) = 0$

$\Rightarrow 4x - 3y = 0$

$\Rightarrow x = \frac{3}{4}y$

Putting $x = \frac{3}{4}y$ in (2) we get $\vec{r} = \dfrac{\frac{3}{4}\hat{i} + \hat{j}}{\sqrt{\frac{9}{16}+1}} = \dfrac{3\hat{i} + 4\hat{j}}{5}$

Thus the required vector is $= \pm \frac{1}{5}(3\hat{i} + 4\hat{j})$.

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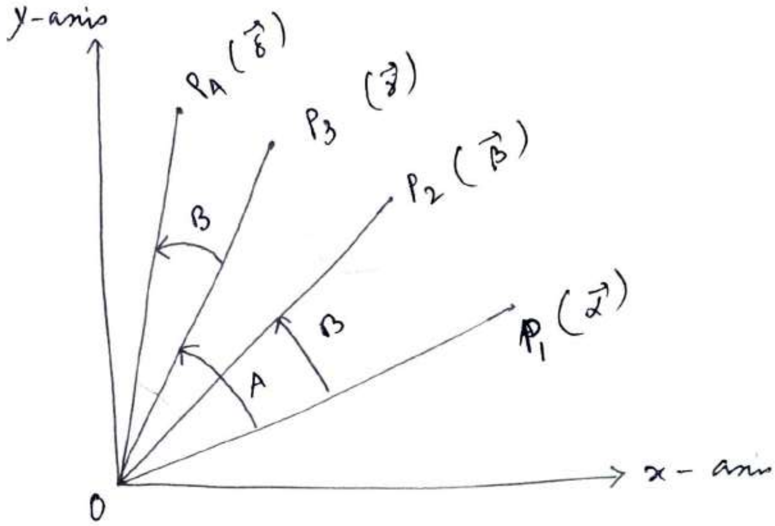
Vector Algebra. (Vector Triple product).

(19) Use $(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}\times\vec{\delta}) + (\vec{\alpha}\times\vec{\gamma})\cdot(\vec{\delta}\times\vec{\beta}) + (\vec{\alpha}\times\vec{\delta})\cdot(\vec{\beta}\times\vec{\gamma}) = 0$ show that

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$.

Solution:



We consider 'O' as origin and two mutually perpendicular lines through 'O' as axes respectively.

We take four pts P_1, P_2, P_3, P_4 in the xy plane s.t

$\angle P_1OP_2 = B$, $\angle P_1OP_3 = A$ and $\angle P_1OP_4 = A+B$.

We also take $\vec{OP_1} = \vec{\alpha}$, $\vec{OP_2} = \vec{\beta}$, $\vec{OP_3} = \vec{\gamma}$ and $\vec{OP_4} = \vec{\delta}$. Then $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ & $\vec{\delta}$ are coplanar vectors lying in the xy plane. Let $\hat{n}$ be ~~the~~ unit perpendicular to ~~the plane~~ the xy-plane. Then from the given result, using the defn of cross product we have

$(|\vec{\alpha}||\vec{\beta}|\sin B\,\hat{n})\cdot(|\vec{\gamma}||\vec{\delta}|\sin B\,\hat{n}) + (|\vec{\alpha}||\vec{\gamma}|\sin A\,\hat{n})\cdot(|\vec{\delta}||\vec{\beta}|\sin(-A)\,\hat{n}) + (|\vec{\alpha}||\vec{\delta}|\sin(A+B)\,\hat{n})\cdot(|\vec{\beta}||\vec{\gamma}|\sin(A-$

$= 0$

$\Rightarrow \sin^2 B - \sin^2 A + \sin(A+B)\sin(A-B) = 0$

$\Rightarrow \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

Again from the given result, we have

$$(\vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}\times\vec{\delta}) + \vec{\alpha}\cdot\{\vec{\gamma}\times(\vec{\delta}\times\vec{\beta})\} + \vec{\alpha}\cdot(\vec{\delta}\times(\vec{\beta}\times\vec{\gamma}))=0$$

$$\Rightarrow (\vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}\times\vec{\delta}) + \vec{\alpha}\cdot\{(\vec{\gamma}\cdot\vec{\beta})\vec{\delta} - (\vec{\gamma}\cdot\vec{\delta})\vec{\beta}\}$$
$$+ \vec{\alpha}\cdot\{(\vec{\delta}\cdot\vec{\gamma})\vec{\beta} - (\vec{\delta}\cdot\vec{\beta})\vec{\gamma}\} = 0.$$

$$\Rightarrow (\vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}\times\vec{\delta}) + (\vec{\alpha}\cdot\vec{\delta})(\vec{\gamma}\cdot\vec{\beta}) - (\vec{\gamma}\cdot\vec{\delta})(\vec{\alpha}\cdot\vec{\beta})$$
$$+ (\vec{\alpha}\cdot\vec{\beta})(\vec{\delta}\cdot\vec{\gamma}) - (\vec{\delta}\cdot\vec{\beta})(\vec{\alpha}\cdot\vec{\gamma}) = 0$$

$$\Rightarrow (\vec{\alpha}\times\vec{\beta})\cdot(\vec{\gamma}\times\vec{\delta}) + (\vec{\alpha}\cdot\vec{\delta})(\vec{\gamma}\cdot\vec{\beta}) - (\vec{\delta}\cdot\vec{\beta})(\vec{\alpha}\cdot\vec{\gamma}) = 0.$$

$$\Rightarrow (|\vec{\alpha}||\vec{\beta}|\sin B\,\hat{n})\cdot(|\vec{\gamma}||\vec{\delta}|\sin B\,\hat{n}) + |\vec{\alpha}||\vec{\delta}|\cos(A+B)\,|\vec{\gamma}||\vec{\beta}|$$
$$\cos(A-B) - |\vec{\delta}||\vec{\beta}|\cos A\,|\vec{\alpha}||\vec{\gamma}|\cos A = 0$$ (using defn of dot product and cross product).

$$\Rightarrow \sin^2 B + \cos(A+B)\cos(A-B) - \cos^2 A = 0.$$

$$\Rightarrow \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B.$$