Optimal Provision of Public goods : We are aware of what Public goods are. We directly proceed to the solution. The technique is simply to hold U_2 at a constant level, $\overline{U2}$, and then maximise U_1 subject to the PPF constraint, denoted by $[F(Q_0, Q_1)]=0$. The problem can be stated as follows:
Maximise $U_1(Q_0, Q^1{}_1)$ subject to a constant $\overline{U2} = U_2(Q_0, Q^2{}_1)$ and a PPF, $F(Q_0, Q_1) = 0$, where
Q_0 denotes quantity of the public good and Q_1 denotes quantity of the private good.
The same amount of the public good Q_0 is consumed by both consumer 1 and 2, i.e., $Q_0 = Q^1{}_0 = Q^2{}_0$
But, $Q_1 = Q^1{}_1 + Q^2{}_1$ for the private good. So we form the Lagrangian as follows:
$\Psi = U_1(Q_0, Q^1{}_1) + \mu(\overline{U2} - U_2(Q_0, Q^2{}_1)) + \xi[F(Q_0, Q_1)]$

The first order conditions are :

$(\delta\Psi/\delta Q_0) = (\delta U_1/\delta Q_0) - \mu(\delta U_2/\delta Q_0) + \xi(\delta F/\delta Q_0) = 0$ .... (1)

$(\delta\Psi/\delta Q^1{}_1) = (\delta U_1/\delta Q^1{}_1) + \xi(\delta F/\delta Q_1) = 0$ .... (2) *[Note that in the PPF, we need not write $\delta Q^1{}_1$, as the opportunity cost in terms of sacrificing some Q_0 in order to produce more Q_1 (the MRT or marginal rate of transformation) will not change if individual 1 consumes that extra Q_1 instead of individual 2].*

$(\delta\Psi/\delta Q^2{}_1) = -\mu(\delta U_2/\delta Q^2{}_1) + \xi(\delta F/\delta Q_1) = 0$ .... (3)

From (3), $\mu = [\xi(\delta F/\delta Q_1)] / (\delta U_2/\delta Q^2{}_1)$. Putting this value in (1), we get:

$(\delta U_1/\delta Q_0) - [\xi(\delta F/\delta Q_1) / (\delta U_2/\delta Q^2{}_1)] . (\delta U_2/\delta Q_0) + \xi(\delta F/\delta Q_0) = 0$

Or, $[\xi(\delta F/\delta Q_1) / (\delta U_2/\delta Q^2{}_1)] . (\delta U_2/\delta Q_0) - (\delta U_1/\delta Q_0) = \xi(\delta F/\delta Q_0)$

Divide both sides by $(\delta U_1/\delta Q^1{}_1)$ to get

$[\xi(\delta F/\delta Q_1) / (\delta U_2/\delta Q^2{}_1)][(\delta U_2/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)] - (\delta U_1/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$

$= \xi(\delta F/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$

Or, $\xi[(\delta F/\delta Q_1)/(\delta U_1/\delta Q^1{}_1)]. [(\delta U_2/\delta Q_0)/(\delta U_2/\delta Q^2{}_1)] - (\delta U_1/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$

$= \xi(\delta F/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$ .... (4)

But from (2), $\xi = -(\delta U_1/\delta Q^1{}_1) / (\delta F/\delta Q_1)$. Putting this value in (4), we get:

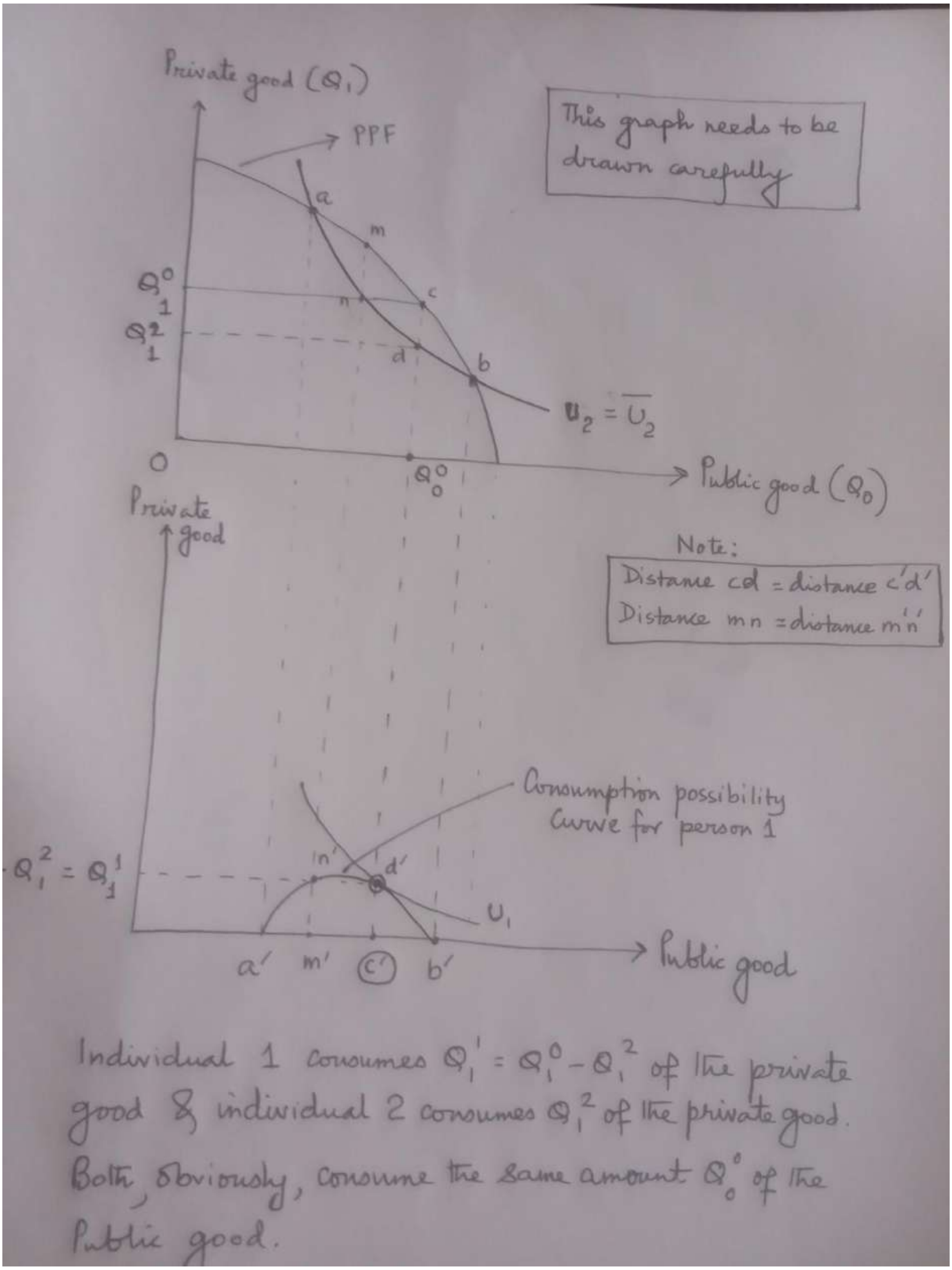
$-\cancel{(\delta U_1/\delta Q^1{}_1)}/(\delta F/\delta Q_1).[(\delta F/\delta Q_1)/\cancel{(\delta U_1/\delta Q^1{}_1)}].[(\delta U_2/\delta Q_0)/(\delta U_2/\delta Q^2{}_1)] - (\delta U_1/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$
$= -(\delta U_1/\delta Q^1{}_1)/(\delta F/\delta Q_1). (\delta F/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)$

All the terms in the above expression carry a minus sign. Let us cancel those first. Then we note that the yellow and ash highlighted terms cancel each other out and the crossed ones also cancel each other out. What remains is:

$[(\delta U_2/\delta Q_0)/(\delta U_2/\delta Q^2{}_1)] + [(\delta U_1/\delta Q_0)/(\delta U_1/\delta Q^1{}_1)] = [(\delta F/\delta Q_0)/(\delta F/\delta Q_1)]$.

That is, $\boxed{\mathbf{MRS^2 + MRS^1 = MRT}}$. This is the famous Samuelson condition for optimal provision of PGs.

We can explain it diagrammatically as well.



In figure 1 (upper panel), we have drawn a PPF and a single U_2 indifference curve, fixing U_2 at $\overline{U2}$. Individual 2 may be placed at any point on this indifference curve subject to the PPF restriction. After he or she is placed at a point, we simply measure the vertical distance between the PPF and that point on the i.c. Say, individual 2 is placed at d, then only cd of the private good is left for individual 1. He (she), however, enjoys the same quantity of public good by definition (OQ^0_0 in this case). In this way, varying points like d, we get the consumption possibility curve for person 1. The optimal allocation is that which gives individual 1 the highest utility (here the curve U_1) subject to $\overline{U2}$ (point d')