On-line Course Materials:

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Subject : **Mathematics**

Year/Semester **: 1st Semester**

Paper: **CC2(theory)**

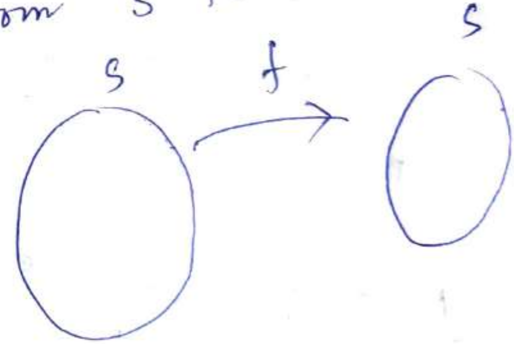
Unit/Chapter /Module : **Unit-2**

Topic/Title: **Some important Problems and its solutions on RELATION (Part-1)**

1. Let S be a non empty finite set. (Ex-3).
Prove that a mapping $f: S \to S$ is injective iff f is surjective.

Solution: Since S contains finite elements, so let $S = \{a_1, a_2, \dots a_n\}$, $n \in \mathbb{N}$.

Let f be any mapping from S to S.



Let f be 1-1 mapping.
We shall show that f is onto.
Since f is injective, so images of different elements of the domain set S are different.
Since S contains n elements, so Images of f contains exactly n elements of the codomain set S. Since codomain set contains exactly n elements, so Image of f = codomain set
$\therefore f$ is surjective.

Conversely, let f be surjective.

We shall show that f is injective.

If possible, let f ~~be~~ not be injective. Then there exist at least two elements in the domain set S s.t their image are same. Since S contains n elements, so Image of f contains atmost $(n-1)$ elements. Since codomain set S contains n elements, so it follows that Image of $f \neq S$.

$\Rightarrow$ f is not surjective

which is a contradiction as f is surjective!

So; f is injective. Hence the result is proved.

Note: If we take infinite set in place of finite set. Is then the statement true?

Ans: NO. For example:

We consider $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) = 2n, \quad \forall n \in \mathbb{N}.$$

Then clearly f is injective but f is not surjective as Range of $f = \{ f(n) \mid n \in \mathbb{N} \}$
$= \{ 2n \mid n \in \mathbb{N} \} =$ set of even positive integers $\neq \mathbb{N} =$ codom

Next we define a mapping $f: \mathbb{N} \to \mathbb{N}$ as (Ex-3) follows

$$f(1) = f(2) = 1$$

$$\& \ f(n) = n-1, \quad \forall n (\geq 3) \in \mathbb{N}$$

Then f is surjective as for any $y \in \mathbb{N}$ (codomain), $y+1 \in \mathbb{N}$ (domain set) and $f(y+1) = y+1-1$
$= y$

But f is not injective as $f(1) = f(2) = 1$

QUES/KROB:

Maps

(2) Give an example of an infinite set S and a mapping $f: S \to S$ such that (ii) f is surjective but not injective:

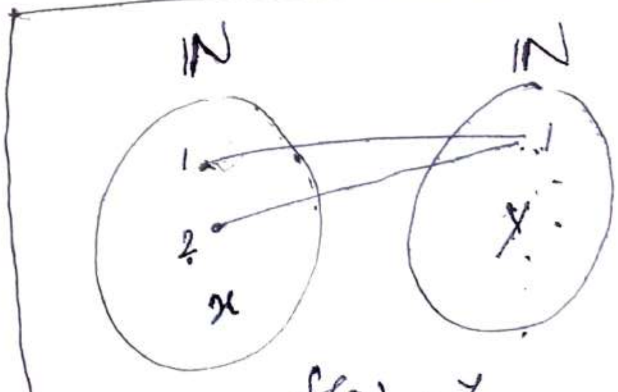
Solution: We consider a mapping $f: \mathbb{N} \to \mathbb{N}$ as follows

$$f(1) = 1 \text{ and } f(x) = x-1, \ \forall x \geq 2$$

We see that $f(1) = 1$ and $f(2) = 1$

Thus $1 \neq 2$ in N (domain) $\Rightarrow f(1) = f(2)$

$\therefore f$ is not injective



Let y be any element of the co-domain set $\mathbb{N}$

$f(x) = y$

$x - 1 = y$ $\quad x = 1 + y$

as possible let $x \in \mathbb{N}$ s.t

Then $1+y \in \mathbb{N}$ (domain) and $f(1+y) = 1+y-1$
$= y$

$\Rightarrow 1+y$ is a pre-image of y. Since y is arbitrary element in $\mathbb{N}$ (codomain set), so each y has a pre-image in the domain set $\mathbb{N}$.

$\therefore$ f is surjective.

Thus f is surjective but not injective.

<u>3(iv)</u> Show that the mapping f is neither injective nor surjective: $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2+1}$, $x \in \mathbb{R}$.

<u>Solution</u>: Let $x_1, x_2 \in \mathbb{R}$ (domain set) s.t $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow x_1 x_2^2 + x_1 = x_1^2 x_2 + x_2$$

$$\Rightarrow x_1 x_2 (x_2 - x_1) - (x_2 - x_1) = 0$$

$$\Rightarrow (x_2 - x_1)(x_1 x_2 - 1) = 0$$

$\Rightarrow$ either $x_2 - x_1 = 0$ or $x_1 x_2 - 1 = 0$

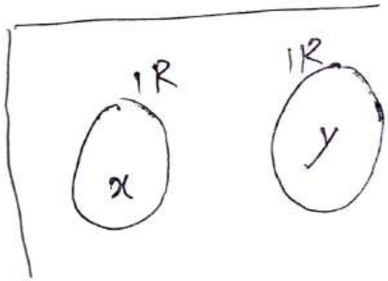
When $x_1 x_2 - 1 = 0$, then $x_1 = \frac{1}{x_2}$, Let $x_2 = 2$

Then $x_1 = \frac{1}{2}$

Then $f(2) = \dfrac{2}{4+1} = 2/5$ and $f\left(\frac{1}{2}\right) = \dfrac{\frac{1}{2}}{\frac{1}{4}+1}$

$= \dfrac{2}{5}$

$\therefore\ 2 \neq \frac{1}{2}$ but $f(2) = f\left(\frac{1}{2}\right)$, so f is not injective.



Let y be any element in the co-domain set $\mathbb{R}$. If possible, let $x \in \mathbb{R}$ s.t $f(x) = y$

$$\Rightarrow \frac{x}{x^2+1} = y$$

$$\Rightarrow x^2 y - x + y = 0$$

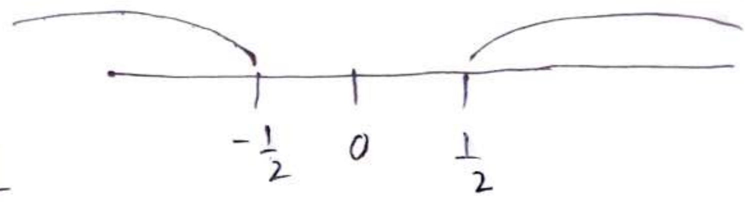
$$\Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y}, \quad (y \neq 0)$$

If $1 - 4y^2 < 0$, then x is imaginary no.

Now $1 - 4y^2 < 0 \Rightarrow 4y^2 > 1$

$\Rightarrow y^2 > \frac{1}{4}$

$\Rightarrow |y| > \frac{1}{2}$



$\Rightarrow$ $y > \frac{1}{2}$ ~~and~~ or $y < -\frac{1}{2}$

Thus x is imaginary no if $y > \frac{1}{2}$ or $y < -\frac{1}{2}$

$\therefore$ when $y > \frac{1}{2}$ and $y < -\frac{1}{2}$, then y has no. pre-image in the domain set $\mathbb{R}$.

$\therefore$ f is not surjective.

Mapping QUES/PROB:

3(ii) Show that the mapping f is neither injective nor surjective : $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$, $\forall x \in \mathbb{R}$

Solution: Since $1, -1 \in \mathbb{R}$ with $1 \neq -1$

but $f(1) = |1| = 1$ and $f(-1) = |-1| = 1$.

$\therefore$ f is not injective.

Let $y \in \mathbb{R}$ (codomain). If possible let $x \in \mathbb{R}$ s.t $f(x) = y$

$\Rightarrow |x| = y$

If we chose $y = -1 < 0$, there for all $x \in \mathbb{R}$, $|x| \geq 0$, so $|x| = y$ is not possible

$\therefore$ $y = -1$ has no pre-image in the domain set $\mathbb{R}$

$\therefore$ f is not surjective

6 (iii) Maps

$f : S \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1-|x|}$, $x \in S$

where $S = \{x \in \mathbb{R} \mid -1 < x < 1\}$

show that f is bijective

Solution: ~~...~~ Let $x, y \in S$. Then we consider three cases

(i) $x, y \geq 0$

(ii) $x, y \leq 0$

(iii) $x \geq 0, \quad y < 0$

Case (i) Let $f(x) = f(y)$. Then this gives

$$\frac{x}{1-|x|} = \frac{y}{1-|y|}$$

or $$\frac{x}{1-x} = \frac{y}{1-y} \quad (\because x, y \geq 0)$$

or $$x - xy = y - xy$$

or $$x = y$$

Thus $f(x) = f(y) \Rightarrow x = y$.

Case (ii) Let $f(x) = f(y)$. Then this gives

$$\frac{x}{1-|x|} = \frac{y}{1-|y|}$$

or $\qquad \dfrac{x}{1+x} = \dfrac{y}{1+y} \quad (\because x, y \leq 0).$

or $\qquad x + xy = y + xy$

or $\qquad x = y$

Thus, $\qquad f(x) = f(y) \Rightarrow x = y.$

Case (iii) Since $x \geq 0$ & $y < 0$, so x & y are two distinct elements of the domain set S. We shall show that $f(x) \neq f(y)$

Now, $f(x) = \dfrac{x}{1-|x|} = \dfrac{x}{1-x}$ $(\because x \geq 0)$

As $0 \leq x < 1$, so $1-x > 0$

$\therefore f(x) \geq 0.$

$f(y) = \dfrac{y}{1-|y|} = \dfrac{y}{1+y}$ $(\because y < 0)$

Since $-1 < y < 0$

$\Rightarrow 1+y > 0$

$\therefore f(y) < 0$ and hence $f(x) \neq f(y)$

Thus $x \neq y \Rightarrow f(x) \neq f(y)$

$\therefore$ from all cases, we conclude that f is injective.

We see that $f(x) = \dfrac{x}{1-|x|}$

$$= \begin{cases} \dfrac{x}{1-x}, & \text{if } x \geq 0 \\ \dfrac{x}{1+x} & \text{if } x < 0 \end{cases}$$

when $0 \le x < 1$, so $\frac{x}{1-x} \geq 0$

& when $-1 < x < 0$, $\frac{x}{1+x} < 0$

$\therefore$ $f(x) \geq 0$ when $0 \le x < 1$ and

$f(x) < 0$ when $-1 < x < 0$.

* Let $y \in \mathbb{R}$. We consider two cases

(i) $y \geq 0$

(ii) $y < 0$.

<u>case (i)</u> Since $y \geq 0$, so $0 \le \frac{y}{1+y} < 1$

$\therefore$ $\frac{y}{1+y} \in S$ and $f\left(\frac{y}{1+y}\right)$

$$= \frac{\frac{y}{1+y}}{1 - \left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}}$$

$= y$

$\therefore \frac{y}{1+y}$ is a pre-image of y

Case (ii) ~~Since~~ Here $y < 0$, Since $-1 < 0$

$\Rightarrow -1 + y < 0 + y$

$\Rightarrow -(1-y) < y$

$\Rightarrow -1 < \frac{y}{1-y}$ ($\because 1-y > 0$ as $y < 0$)

Thus $-1 < \frac{y}{1-y} < 0$ and hence

$\frac{y}{1-y} \in S$.

Now, $f\left(\frac{y}{1-y}\right) = \dfrac{\frac{y}{1-y}}{1 - \left|\frac{y}{1-y}\right|}$

$= \dfrac{\frac{y}{1-y}}{1 + \frac{y}{1-y}}$

$= y$

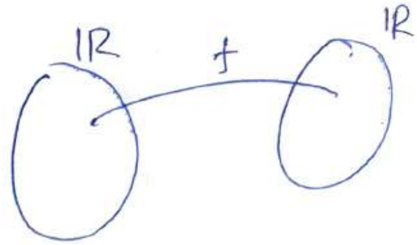
Thus $\frac{y}{1-y}$ is a pre-image of y.

$\therefore$ by both cases, for any $y \in \mathbb{R}$, y has a pre-image in the domain set S.

$\therefore f$ is surjective

3) (iii) Show that the mapping f is neither injective nor surjective.

$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2 + x, \ \forall x \in \mathbb{R}$



Solution: Let $x_1, x_2 \in \mathbb{R}$ (domain set) s.t

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^2 + x_1 = x_2^2 + x_2$$

$$\Rightarrow x_1^2 - x_2^2 + x_1 - x_2 = 0$$

$$\Rightarrow (x_1 + x_2)(x_1 - x_2) + (x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad \text{or} \quad x_1 + x_2 + 1 = 0$$

$$\Rightarrow x_1 = x_2 \quad \text{or} \quad x_1 = -1 - x_2$$

Thus we can say that $f(x_1) = f(x_2)$ when $x_1 = -1 - x_2$

Choosing $x_2 = 0$, we get $x_1 = -1$

Thus $0 \neq -1$ but $f(0) = 0 = f(-1)$

$\therefore$ f is not injective.

Let y be any element in $\mathbb{R}$ (codomain set)

If possible, let x be a pre-image of y.

Then $f(x) = y$

$$\Rightarrow x^2 + x - y = 0$$

$$\therefore x = \frac{-1 \pm \sqrt{1+4y}}{2}$$

But when $1+4y < 0$ i.e $y < -\frac{1}{4}$, then x is imaginary. Thus for all y in the codomain set $\mathbb{R}$ s.t $y < -\frac{1}{4}$ has no pre-image in the domain set $\mathbb{R}$.

So, f is not surjective.

(iv) $f: \mathbb{R} \to \mathbb{R}$ s.t $f(x) = \frac{x}{x^2+1}$. $x \in \mathbb{R}$.

Same as 3(iii)

5. (i) Show that the mapping f is surjective but not injective

$f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = \left[\frac{n+1}{2}\right]$

$\forall n \in \mathbb{N}$.

Solution:

Here, $f(1) = \left[\frac{1+1}{2}\right] = 1$

$f(2) = \left[\frac{2+1}{2}\right] = 1.$

Thus $f(1) = f(2)$ where $1 \neq 2$

$\therefore$ f is not injective.

Let y be any element of the co-domain set $\mathbb{N}$.

Then $2y \in \mathbb{N}$ (domain set).

Now $f(2y) = \left[\frac{2y+1}{2}\right] = \left[y + \frac{1}{2}\right] = y$

Thus $2y$ is a pre-image of y. Since y is arbitrary, so each element of co-domain set has a pre-image in the domain set, so f is surjective

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4(iii) Show that the mapping f is injective but not surjective.

$f : \mathbb{Z} \to \mathbb{Q}$ defined by $f(x) = 2^x, x \in \mathbb{Z}$

Solution: Let $x_1, x_2 \in \mathbb{Z}$ s.t $f(x_1) = f(x_2)$

Then $f(x_1) = f(x_2) \Rightarrow 2^{x_1} = 2^{x_2}$

$\Rightarrow x_1 = x_2$

$\therefore$ f is injective.

Let y be any element of $\mathbb{Q}$.

If possible, let x be a pre-image of y.

Then $f(x) = y$

$\Rightarrow 2^x = y$

$\Rightarrow x = \dfrac{\log y}{\log 2}$

But we know that $\log y$ is defined only when $y > 0$. Since $y \in \mathbb{Q}$, so when we take $y < 0$ in $\mathbb{Q}$; then there exist no x in $\mathbb{Z}$ s.t $f(x) = y$ as $\log y$ is undefined. So f is not surjective.

5 (iii) Show that the mapping f is surjective but not injective.

$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$, $x \in \mathbb{R}$.

Solution: Let $x_1, x_2 \in \mathbb{R}$ (domain set) s.t

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^3 - x_1 = x_2^3 - x_2$$

$$\Rightarrow x_1^3 - x_2^3 - x_1 + x_2 = 0$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) - (x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2 - 1) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ or } x_1^2 + x_1 x_2 + x_2^2 - 1 = 0$$

If $x_1^2 + x_1 x_2 + x_2^2 - 1 = 0$, and we choose $x_1 = 1$, then $x_2 + x_2^2 = 0$

$$\Rightarrow x_2(\cancel{x_2}\, 1 + x_2) = 0$$

$$\Rightarrow x_2 = 0 \text{ or } x_2 = -1$$

Then $f(0) = 0 = f(-1)$, so f is not injective.

Let y be any element of the co-domain set $\mathbb{R}$. ~~Hence~~ If possible, let x be a pre-image of y. Then $f(x) = y$

$$\Rightarrow x^3 - x = y$$

$$\Rightarrow x^3 - x - y = 0$$

which is cubic eqn in x, so it has at least one real root, say x_1

Then $x_1 \in \mathbb{R}$ and $x_1^3 - x_1 - y = 0$.

Now $f(x_1) = x_1^3 - x_1$

$= y \quad (\because x_1^3 - x_1 - y = 0)$

$\Rightarrow x_1$ is a pre-image of y. Since y is arbitrary element of $\mathbb{R}$, so each element of $\mathbb{R}$ has a pre-image in the domain set $\mathbb{R}$. So f is surjective.

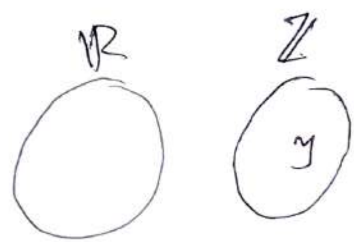
5 Show that the mapping f is surjective but not injective. (ii) $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = [x]$, $x \in \mathbb{R}$.

Solution: $f(1) = [1] = 1$ & $f(1.1) = [1.1] = 1$

Thus we get $1, 1.1 \in \mathbb{R}$ with $1 \neq 1.1$

but $f(1) = f(1.1)$.

$\therefore$ f is not 1-1 mapping.



Let $y \in \mathbb{Z}$. Then $y \in \mathbb{R}$ and

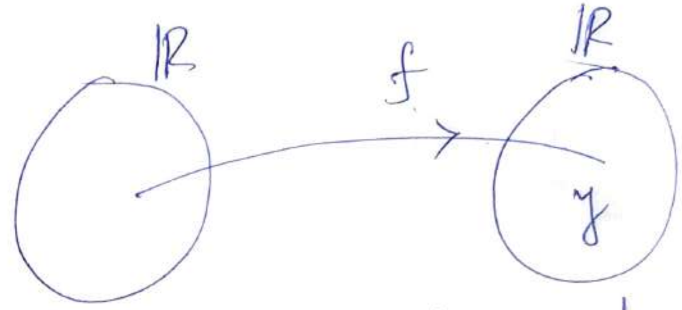
$f(y) = [y] = y$.

this shows that y is a pre-image of y

Since y is any element of $\mathbb{Z}$, so each element of $\mathbb{Z}$ has a pre-image in the domain set $\mathbb{R}$. $\therefore$ f is onto mapping.

Show that the mapping $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3, \ \forall x \in \mathbb{R}$ is bijective and find f^{-1}

Solution:



Let $x_1, x_2 \in \mathbb{R}$ (domain set) s.t

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0 \quad \text{--- (1)}$$

If $x_1^2 + x_1 x_2 + x_2^2 = 0$, then

$$x_1 = \frac{-x_2 \pm \sqrt{x_2^2 - 4x_2^2}}{2} = \frac{-x_2 \pm \sqrt{-3x_2^2}}{2}$$

If $x_2 = 0$, then $x_1 = 0$. If x_2 is non zero real no, then x_1 is imaginary no.

Since $x_1, x_2 \in \mathbb{R}$, so $x_1^2 + x_1x_2 + x_2^2 = 0$ only when $x_1 = x_2 = 0$.

Then from (1), we get $x_1 - x_2 = 0$ or $x_1^2 + x_1x_2 + x_2^2 = 0$

$\Rightarrow x_1 = x_2$ or $x_1 = x_2 = 0$

Thus $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is injective mapping.

Let y be any element of $\mathbb{R}$ (codomain set)

Then we consider the eqn $t^3 - y = 0$ — (1)

Since degree of (1) is $3 =$ odd, so $\exists$ at least one real root. ~~the~~ the eqn (1) has at least one real root. Let $x_1 (\in \mathbb{R})$ be ~~the~~ a root of (1). Then

$x_1^3 - y = 0$.

Now, $f(x_1) = x_1^3 = y$ $(\because x_1^3 - y = 0)$

$\therefore$ x_1 is a pre-image of y in $\mathbb{R}$ (domain set).

Since y is arbitrary element of $\mathbb{R}$ (codomain set)

So each element of codomain set has a pre-image in domain set. $\therefore$ f is onto mapping.

Thus f is bijective, and hence f^{-1} exist.

Now, $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is 

$$f^{-1}(y) = y^{1/3}, \quad \forall y \in \mathbb{R}$$

$x^3 = y$

$x = y^{\frac{1}{3}}$

$f^{-1}(x^3) = x$

$f^{-1}(y) =$