On-line Course Materials:

Prepared by: Masiur Rahaman Sardar
Assistant Professor
Dept. Of Mathematics
City College
102/1 Raja Rammohan Roy Sarani, Kolkata -09
Mobile : 9830458374
Email :sardarmasiur@gmail.com

Subject : **Mathematics**

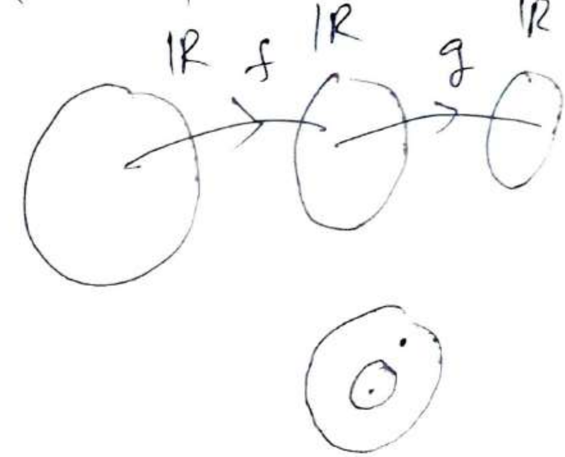
Year/Semester **: 1st Semester**

Paper: **CC2(theory)**

Unit/Chapter /Module : **Unit-2**

Topic/Title: **Some important Problems on Mapping and its solutions (Part-2)**

7 (iii) Find $g \circ f$ and $f \circ g$ if $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x| + x$, $x \in \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = |x| - x$, $x \in \mathbb{R}$.



__Solution__ Clearly $g \circ f: \mathbb{R} \to \mathbb{R}$ and $f \circ g: \mathbb{R} \to \mathbb{R}$ are defined.

Let $x \in \mathbb{R}$.

then $(g \circ f)(x) = g(f(x))$

$= g(|x| + x)$

$= g(y), \quad y = |x| + x$

$= |y| - y$

$= \big| |x| + x \big| - (|x| + x)$

$= 0, \quad \forall x \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x))$$

$$= f(|x| - x)$$

$$= f(y), \quad y = |x| - x$$

$$= |y| + y$$

$$= \big| |x| - x \big| + (|x| - x)$$

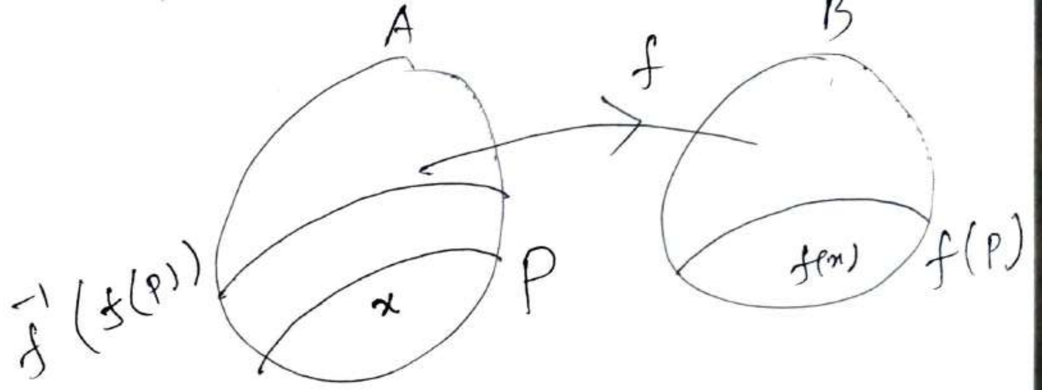
$$= \begin{cases} 0, & \text{if } x \geq 0 \\ 4z & \text{if } x = -z < 0 \end{cases}$$

12: Let $f: A \to B$ and $P \subset A$, Prove that

(i) $P \subseteq f^{-1}(f(P))$ (ii) $P = f^{-1}(f(P))$ if f be injective

Give an example where $P \neq f^{-1}(f(P))$.

Solution: (i)



Let $x \in P$. Then $f(x) \in f(P)$

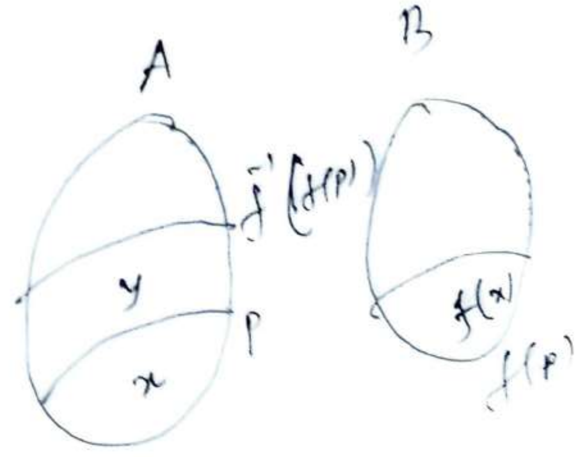
$\Rightarrow x \in f^{-1}(f(P))$

$\therefore P \subseteq f^{-1}(f(P))$.

(ii) Here f is injective.

In this case we also get

$$P \subseteq f^{-1}(f(P)) \quad \text{— (i)}$$



Let $y \in f^{-1}(f(P))$

Then $f(y) \in f(P)$

$\Rightarrow f(y) = f(x)$, for some $x \in P$

$\Rightarrow y = x$ (as f is injective)

$\Rightarrow y = x \in P$

Thus $y \in f^{-1}(f(P)) \Rightarrow y \in P$

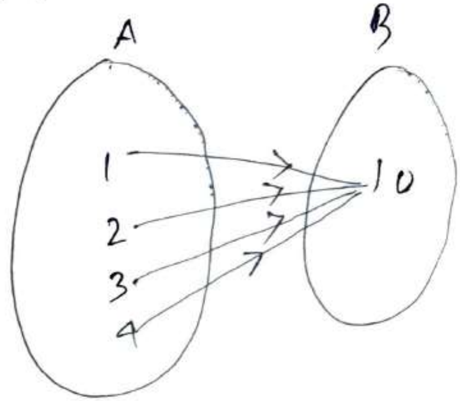
$\therefore f^{-1}(f(P)) \subseteq P$ — (ii)

Then from (i) & (ii) we get

$$P = f^{-1}(f(P)).$$

Counter example: Let $f : A = \{1, 2, 3, 4\} \rightarrow B = \{10\}$

be defined by $f(1) = f(2) = f(3) = f(4) = 10$.



Let $P = \{3, 4\}$. Then $P \subset A$.

Then $f(P) = \{ \cancel{10} \; f(x) \mid x \in P \}$

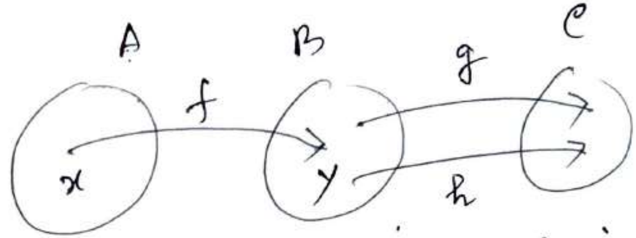
$= \{10\}$

Now, $f^{-1}(f(P)) = f^{-1}(\{10\}) = \{1, 2, 3, 4\}$

$\therefore \; P \neq f^{-1}(f(P))$.

S.K. Mapa

(8) Let $f : A \to B$, $g : B \to C$, $h : B \to C$ be mappings such that $g \circ f = h \circ f$ and f is surjective. Prove that $g = h$



Solution:

Let y be any element of B. Since $f : A \to B$ is onto mapping, so there exist $x \in A$ s.t

$f(x) = y$ ---(1)

Since $g \circ f = h \circ f$, so we get

$$(g \circ f)(x) = (h \circ f)(x)$$

or $$g(f(x)) = h(f(x))$$

or $$g(y) = h(y)$$

Since y is arbitrary element of B, so

$$g(y) = h(y), \quad \forall y \in B$$

so, $g = h$.

(13) Let A be a set of n elements and B be a set of m elements. Show that (i) The total no of mapping from A to B is m^n (ii) If $n \le m$, the total no of injective mappings from A to B is $\dfrac{m!}{(m-n)!}$.

<u>Solution</u> (i) Let $A = \{a_1, a_2, \ldots a_n\}$ & $B = \{b_1, b_2 \ldots b_m\}$. Since image of a_1 must be either b_1 or b_2 or $b_3 \ldots$ or b_m, so there are m choice for image of a_1. Simily there are m choice for images of $a_2, a_3, \ldots a_n$. So the total no of mappings from A to B is $m.m.m \ldots m$ (n times) $= m^n$

(ii) Since for any injective mapping images are different for different elements, so there are m choice for image of a_1, $(m-1)$ choice for image of a_2, $(m-2)$ choice for image of $a_3, \ldots$ $(m-(n-1))$ choice for image of a_n. Thus total no of injective mapping from A to B is

$$m(m-1)(m-2)\ldots(m-(n-1))$$

$$= \frac{m(m-1)(m-2)\ldots(m-(n-1))(m-n)(m-(n+1))\ldots 1}{(m-n)(m-(n+1))\ldots 1} = \frac{m!}{(m-n)!}$$

6(b)

Problem: Let $f : A \longrightarrow B$ and $C, D \subseteq A$.
Then $f(C \cap D) = f(C) \cap f(D)$.

Let $x \in f(C \cap D)$. Then there exist $y \in C \cap D$
s.t $f(y) = x$.

Now, $y \in C \cap D \Rightarrow y \in C$ and $y \in D$.

$\Rightarrow f(y) \in f(C)$ and $f(y) \in f(D)$

$\Rightarrow f(y) \in f(C) \cap f(D)$

$\Rightarrow x \in f(C) \cap f(D)$

Thus $f(C \cap D) \subseteq f(C) \cap f(D)$ --- (1)

Again let $z \in f(C) \cap f(D)$. Then
$z \in f(C)$ and $z \in f(D)$. Then there exist
two elements $x_1 \in C$ and $x_2 \in D$ s.t
$f(x_1) = z$ and $f(x_2) = z$

$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ ($\because f$ is injective)

Thus $x_1 = x_2 = x$ (say) $\in C$ & D ie
$x \in C \cap D \Rightarrow f(x) \in f(C \cap D)$

$\Rightarrow z \in f(C \cap D)$

Thus $f(C) \cap f(D) \subseteq f(C \cap D)$ ... (2). From
(1) & (2) we have $f(C \cap D) = f(C) \cap f(D)$.

6 | (ix) Show that f is bijective mapping and determine f^{-1} where $f: \mathbb{R}-\{1\} \Rightarrow \mathbb{R}-\{1\}$.

with $f(x) = \frac{x+1}{x-1}$, $x \in \mathbb{R}-\{1\}$.

Solution: Let $x_1, x_2 \in \mathbb{R}-\{1\}$ s.t

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}$$

$$\Rightarrow x_1x_2 - x_1 + x_2 - 1 = x_1x_2 + x_1 - x_2 - 1$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ $f: \mathbb{R}-\{1\} \longrightarrow \mathbb{R}-\{1\}$ is an injective mapping.

Let y be any element of $\mathbb{R}-\{1\}$.

Then $y \neq 1$. So $\frac{1+y}{y-1}$ is a real no.

Clearly $\frac{1+y}{y-1} \neq 1$, if not, $1+y = y-1$

$\Rightarrow 2 = 0$

which is a contradiction

So $\dfrac{1+y}{y-1}$ is real no not equal to 1.

$$\therefore \quad \frac{1+y}{y-1} \in \mathbb{R} \sim \{1\}$$

Now, $$f\left(\frac{1+y}{y-1}\right) = \frac{1 + \frac{1+y}{y-1}}{\frac{1+y}{y-1} - 1}$$

$$= \frac{2y}{2} = y.$$

Thus $\dfrac{1+y}{y-1}$ is a pre-image of y.

Since y is arbitrary, so each element of the co-domain set $\mathbb{R}\sim\{1\}$ has a pre-image in the domain set $\mathbb{R}\sim\{1\}$. So f is surjective and hence f is bijective.

2nd: From defn of inverse,
$$f^{-1} : \mathbb{R}\sim\{1\} \longrightarrow \mathbb{R}\setminus\{1\} \quad \text{where}$$

$$f^{-1}\left(\frac{x+1}{x-1}\right) = x$$

$$\text{or } f^{-1}(y) = \frac{y+1}{y-1}, \quad \forall y \in \mathbb{R}\setminus\{1\}.$$

9. Let $g : A \to B$, $h : A \to B$, $f : B \to C$ be mappings s.t $f \circ g = f \circ h$ and f is 1-1. Prove that $g = h$

Let x be any element of A

Since $f \circ g = f \circ h$

$\therefore (f \circ g)(x) = (f \circ h)(x)$

$\Rightarrow f(g(x)) = f(h(x))$

$\Rightarrow g(x) = h(x)$ ($\because f$ is 1-1)

Since x is arbitrary in A,

so $g(x) = h(x)$, $\forall x \in A$

$\Rightarrow g = h$. (since domain & codomain set of g & h are identical)

11. Let $f : A \to B$ be a mapping. A relation ρ is defined on A by "$x \rho y$ iff $f(x) = f(y)$ $x, y \in A$". Show that ρ is equivalence relation.

Solution: Let $x \in A$.

Then $f(x) = f(x), \quad \forall x \in A$

$\Rightarrow x \rho x$ holds, $\forall x \in A$

$\Rightarrow \rho$ is reflexive relation on A.

Let $x, y \in A$ s.t $x \rho y$

Then $x \rho y \Rightarrow f(x) = f(y)$

$\Rightarrow f(y) = f(x)$

$\Rightarrow y \rho x$

$\therefore \rho$ is symmetric

Let $x, y, z \in A$ s.t $x \rho y$ & $y \rho z$.

Then $x \rho y \Rightarrow f(x) = f(y)$ and

$y \rho z \Rightarrow f(y) = f(z)$

$\Rightarrow f(x) = f(z)$

$\Rightarrow x \rho z$.

$\therefore \rho$ is transitive.

Thus ρ is an equivalence relation.

C. H - 2007 Define Injective mapping. A mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $f(m, n) = 2^m 3^n$. Show that f is injective but not surjective. If A & B be two sets having n-distinct elements, show that number of bijective mappings from A to B is $n!$

Solution: Injective mapping: Let $f : A \to B$ be a mapping. Then f is said to be injective mapping if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, $x_1, x_2 \in A$.

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{N} \times \mathbb{N}$ s.t

$$f(x_1, y_1) = f(x_2, y_2)$$

$$\Rightarrow 2^{x_1} 3^{y_1} = 2^{x_2} 3^{y_2}$$

$$\Rightarrow 2^{x_1 - x_2} = 3^{y_2 - y_1} \quad \text{—— (1)}$$

But (1) true only when $x_1 - x_2 = 0$ & $y_2 - y_1 = 0$

$$\Rightarrow x_1 = x_2 \ \& \ y_1 = y_2$$

$$\Rightarrow (x_1, y_1) = (x_2, y_2).$$

$\therefore$ f is injective mapping

~~Clearly~~ Since $2 \cdot 5 = 10 \in \mathbb{N}$ and 10 can not be expressed as $2^m 3^n$ for some positive integers m & n. So 10 has no pre-image in the domain set $\mathbb{N} \times \mathbb{N}$. So, f is not surjective mapping.

Last portion: Let $A = \{a_1, a_2, \dots a_n\}$ & $B = \{b_1, b_2, \dots b_n\}$

Then for a bijective mapping $f : A \to B$, we can define f as follows.

$f(a_1)$ goes to any element of B, so there are n choice

Then $f(a_2)$ goes to any of $(n-1)$ element of B, so there are $(n-1)$ choice.

Thus total choice for $f(a_1)$ & $f(a_2)$ is $n.(n-1)$.

proceeding this way, we get the total choice for $f(a_1), f(a_2) \dots f(a_n)$ is $n.(n-1).(n-2) \dots 1$.

i.e. $n!$

Thus total no of bijective mapping from A to B is $n!$