2020

MATHEMATICS — HONOURS

Paper : CC-1

Unit : 1, 2, 3

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the following multiple choice questions. Each question has four possible answers, of which exactly one is correct. Choose the correct option and justify your answer. (1+1)×10

(a) If $\lim\limits_{x\to 0}\dfrac{\sin 2x + p\sin x}{x^3}$ is finite, then the value of p is

(i) -2 (ii) -1 (iii) 1 (iv) 0.

(b) If $y = 2\cos x(\sin x - \cos x)$, then the value of $y_{20}(0)$ is [$y_{20}(0)$ denotes the 20th derivative of y at $x = 0$]

(i) -2^{20} (ii) 2^{20} (iii) 2^{-20} (iv) 2^{10}.

(c) The curvature of the curve $y = f(x)$ is zero at every point on the curve. Which one of the following could be $f(x)$?

(i) $ax + b$ (ii) $ax^2 + bx + c$ (iii) $\sin x$ (iv) $\cos x$.

(d) The curve $y = e^{2020x}$ is

(i) convex with respect to the y-axis (ii) convex with respect to the x-axis

(iii) concave with respect to the y-axis (iv) concave with respect to the x-axis.

(e) $r = \dfrac{5}{2}\sec^2\dfrac{\theta}{2}$ is the polar equation of

(i) an ellipse (ii) a straight line (iii) a parabola (iv) a circle.

(f) The equation of the plane passing through the points (4, 3, 1) and (1, –3, 4) and parallel to the y-axis is

(i) $x - z + 5 = 0$ (ii) $x + z - 5 = 0$ (iii) $x - z - 5 = 0$ (iv) $x + z + 5 = 0$.

(g) The radius of the sphere $3x^2 + 3y^2 + 3z^2 + 2x - 4y - 2z - 1 = 0$ is

(i) 1 unit (ii) 2 units (iii) 4 units (iv) 6 units.

(h) The values of 'a' and 'd' for which the straight line

$$\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z+3}{3}$$

lies on the plane $ax + 3y - 5z + d = 0$ are respectively

(i) 2, 23 (ii) 9, –30 (iii) –9, 30 (iv) 2, –23

(i) The angle between the planes $\vec{r}\cdot\left(2\hat{i}-\hat{j}+2\hat{k}\right)=6$ and $\vec{r}\cdot\left(3\hat{i}+6\hat{j}-2\hat{k}\right)=9$ is

(i) $\cos^{-1}\left(\frac{4}{21}\right)$ (ii) $\sin^{-1}\left(\frac{4}{21}\right)$ (iii) $\cos^{-1}\left(\frac{-4}{21}\right)$ (iv) $\sin^{-1}\left(\frac{-4}{21}\right)$

(j) A particle moves along a curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where 't' is time. Then the velocity of the particle at $t = \pi$ is

(i) $e^{\pi}\hat{i}-6\hat{k}$ (ii) $-e^{\pi}\hat{i}-6\hat{k}$ (iii) $-e^{-\pi}\hat{i}+6\hat{k}$ (iv) $-e^{-\pi}\hat{i}-6\hat{k}$.

2. Answer ***any three*** questions :

(a) If $y=e^{m\sin^{-1}x}$, show that (i) $\left(1-x^2\right)y_{n+2}-(2n+1)xy_{n+1}-(m^2+n^2)y_n=0$, where $y_n=\frac{d^n y}{dx^n}$ and

(ii) also find y_n when $x = 0$. 3+2

(b) Prove that the envelope of the parabolas which touch the coordinate axes at $(\alpha, 0)$ and $(0, \beta)$, where α, β are connected by $\alpha + \beta = c$, is

$$x^{1/3}+y^{1/3}=c^{1/3},\ \text{where } c \text{ is a constant.}$$ 5

(c) Find the rectilinear asymptotes of the curve $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 = 2$. 5

(d) If $I_n=\int_0^1 x^n \tan^{-1}x\,dx$, show that $(n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\frac{1}{n}$. 5

(e) Find the length of the perimeter of the curve $r = 2(1 - \cos\theta)$. 5

3. Answer ***any four*** questions : 5×4

(a) Show that the triangle formed by the pole and the points of intersection of the circle $r = 4\cos\theta$ with the line $r\cos\theta = 3$ is equilateral.

(b) A chord PQ of a conic whose eccentricity is e and semi-latus rectum l subtends a right angle at the focus S. Prove that

$$\left(\frac{1}{SP}-\frac{1}{l}\right)^2+\left(\frac{1}{SQ}-\frac{1}{l}\right)^2=\frac{e^2}{l^2}.$$

(c) A square ABCD of diagonal $2a$ is folded along the diagonal AC so that the planes DAC and BAC are at right angles. Find the shortest distance between AB and DC.

(d) Find the equation of the plane containing the line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$.

(e) Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ is the ellipse $x^2+y^2+z^2=a^2+b^2$.

(f) Find the equation of the right circular cylinder of radius 2 whose axis is the straight line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{2}$.

(g) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes at A, B, C. Find the equation of the cone generated by the straight lines drawn from O to meet the circle through ABC.

4. Answer ***any two*** questions : 5×2

(a) If $\left[\vec{a}\,\vec{b}\,\vec{c}\right]\neq 0$, prove that any vector $\vec{d}$ can be expressed as

$$\vec{d}=\frac{\vec{d}.\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{b}\times\vec{c}+\frac{\vec{d}.\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{c}\times\vec{a}+\frac{\vec{d}.\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}\vec{a}\times\vec{b}.$$

(b) Evaluate $\int_C \vec{F}.d\vec{r}$, where C consists of a part of the x-axis from $x = 2$ to $x = 4$ and then the portion of the line $x = 4$ from $y = 0$ to $y = 12$, where $\vec{F}=xy\hat{i}+(x^2+y^2)\hat{j}$.

(c) A rigid body is spinning with an angular velocity 5 radians per second about an axis parallel to $\hat{i}+\hat{j}+\hat{k}$ and passing through the point $\hat{i}+2\hat{j}-\hat{k}$. Find the velocity of the particle at the point $2\hat{i}+\hat{j}-\hat{k}$.