# 2023

## PHYSICS — HONOURS

### Paper : DSE-A1.1 and DSE-A-1.2

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

### Paper : DSE-A1.1

### (Advanced Mathematical Methods)

### Full Marks : 65

Answer ***question nos.*** 1 and 2, and ***any four*** questions from the rest (***Q.*** 3 to ***Q.*** 8).

1. Answer ***any five*** questions : 2×5

(a) A binary operation is defined as : $a * b = a^2 + b^2$ $(a, b \in \mathbb{R})$. Check commutativity and associativity of the operation.

(b) Find out if $W$ is a subspace of vector space $R^n$, $(n \geq 3)$, where $W = \{\alpha : \alpha \in R^n \text{ and } a_1 + 3a_2 = a_3\}$.

(c) If $P_1$ and $P_2$ are two projection operators, then under what condition is $P_1 + P_2$ also a projection operator?

(d) Show that $\vec{\nabla} \times \vec{\nabla}\phi = 0$ using index notation, where $\phi$ is a scalar field.

(e) Show that a 2nd rank covariant symmetric tensor remains symmetric under a general coordinate transformation.

(f) For what real values of $k$ does the set $S$ form a basis of $\mathbb{R}^3$ :

$$S = \{(k, 0, 1), (1, k+1, 1), (1, 1, 1)\}.$$

(g) Define equivalent representations in group theory.

2. Answer ***any three*** questions :

(a) Find out if the following mappings are linear transformation (homomorphism) or not.

(i) $F : R^2 \rightarrow R^2$ is defined by $F(x, y) = (x + y, x)$.

(ii) $F : R^3 \rightarrow R^2$ is defined by $F(x, y, z) = (|x|, y + z)$. 2+3

(b) Find $\cos\theta$, where $\theta$ is the angle between :

(i) $u = (1, 3, -5, 4)$ and $v = (2, -3, 4, 1)$ in $R^4$.

(ii) $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, where $\langle A|B\rangle = tr\left(B^T A\right)$. 2+3

(c) Write down the general expression for moment of inertia $I_{ij}$. Show that $I_{ij}$ transforms as a 2nd rank tensor under rotation in $R^3$. 2+3

(d) (i) Show that all $(n \times n)$ unitary matrices form a group under multiplication.

(ii) Show that all such matrices with determinant 1 form a subgroup. 3+2

(e) Suppose $\{e, a, b\}$ forms a group under multiplication, where $e$ is the identity element. Construct the group multiplication table. Find the inverse element of $a$ and $b$. 4+1

**3.** (a) Show that a set of orthogonal vectors is linearly independent.

(b) Find a unit vector orthogonal to the vectors $\alpha_1 = (0, 2, 1)$ and $\alpha_2 = (3, 1, 2)$.

(c) (i) A linear transformation $T$ in $\mathbb{R}^2$ is defined as $T|e_1\rangle = |e_2\rangle$ and $T|e_2\rangle = -|e_1\rangle$. Write down the matrix representation of $T$ in $\{|e_1\rangle, |e_2\rangle\}$ basis.

(ii) Let $|\alpha_1\rangle = |e_1\rangle + |e_2\rangle$ and $|\alpha_2\rangle = -|e_1\rangle$ be another basis. Write down the matrix representation of $T$ relative to $\{|\alpha_1\rangle, |\alpha_2\rangle\}$ basis. 3+2+(2+3)

**4.** (a) Let $V = P_2(t)$ with inner product $\langle f | g\rangle = \int_0^1 f(t)g(t)dt$.

(i) Find $\langle f | g\rangle$, where $f(t) = t + 2$ and $g(t) = t^2 - 3t + 4$.

(ii) Find the matrix $A$ of the inner product with respect to the basis $\{1, t, t^2\}$ of $V$, where $A_{ij} = \langle e_i | e_j\rangle$, $e_i$'s are elements of $V$.

(b) Let $U$ be the subspace of $R^4$ spanned by $v_1 = (1, 1, 1, 1)$, $v_2 = (1, -1, 2, 2)$ and $v_3 = (1, 2, -3, -4)$.

(i) Apply Gram-Schmidt algorithm to find an orthogonal and orthonormal basis for $U$.

(ii) Find the projection of $v = (1, 2, -3, 4)$ onto $U$. (2+2)+(4+2)

**5.** (a) Derive the metric tensor for 3D spherical polar coordinate.

(b) Write the transformation rule for mixed tensor of rank two.

(c) Using the expressions of Lorentz transformation, obtain the Lorentz transformation matrix. At which limit, the Lorentz transformation becomes similar to Galilean transformation? 3+2+(4+1)

**6.** Maxwell's equations in covariant form is represented as $\partial_\mu F^{\mu\nu} = j^\nu$.

(a) Show that $\partial_\nu j^\nu = 0$ using antisymmetry of $F^{\mu\nu}$.

(b) Given $F^{0i} = E^i$, $(i = 1, 2, 3)$ and $F^{ij} = \epsilon^{ijk}B_k$, construct the $F^{\mu\nu}$ matrix in $E^i$'s and $B^j$'s. From inhomogeneous Maxwell's equations $\partial_\mu F^{\mu\nu} = j^\nu$, show that its zeroth component yields $\nabla.\vec{E} = \frac{\rho}{\epsilon_0}$.

(c) Construct $F_{\mu\nu}$ from $F^{\mu\nu}$. (use the signature $\{1, -1, -1, -1\}$) and calculate $F_{\mu\nu}F^{\mu\nu}$.

2+(2+2)+(2+2)

7. (a) Construct the group multiplication table for the set of elements $\{1, i, -1, -i\}$. Find out the identity element.

(b) Consider $H = \{1, -1\}$. Show that they form a subgroup of the above group.

(c) For group $G$ and $G'$, we define group homomorphism $\phi : G \to G'$ as $\phi(ab) = \phi(a)\phi(b)$. If $e$ and $e'$ be the group identity elements of $G$ and $G'$ respectively, show that

(i) $\phi(e) = e'$. (ii) $\phi(a^{-1}) = (\phi(a))^{-1}$. (3+1)+2+(2+2)

8. (a) Given a Lie algebra of matrices $X_a$, $X_b$ and $X_c$ via structure constant $f_{abc}$ : $[X_a, X_b] = i\, f_{abc} X_c$, where $f_{abc}$'s are real.

Show that

(i) $f_{abc} = -f_{bac}$

(ii) $-X_a^\dagger$ matrices also satisfy same Lie algebra.

(b) In SO(3) group Rotations about $X$, $Y$ and $Z$-axes are represented respectively as

$$R_X(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, R_Y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix} \text{ and } R_Z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Show that rotations about $X$-axis form a subgroup of SO(3). (Same is true for $Y$ and $Z$-axis).

(ii) Find the associated generators $T_X$, $T_Y$ and $T_Z$. (2+2)+(3+3)

## Paper : DSE-A1.2

## (Laser and Fibre Optics)

**Full Marks : 65**

Answer ***question nos.*** 1 and 2, and ***any four*** questions from the rest.

**1.** Answer ***any five*** questions : 2×5

(a) Determine the emission frequency required to have a temporal coherence length of 10 m by a source having wavelength of 488 nm.

(b) The ratio of population of two energy levels is $1.059 \times 10^{-30}$. Find the wavelength of Laser emitted at 330 K.

(c) What is metastable state? What is the importance of such state in achieving laser action?

(d) Draw the schematic diagram of $TEM_{10}$ and $TEM_{22}$ mode.

(e) Discuss the role of reflectors in an optical resonator.

(f) Distinguish between holography and photography.

(g) What is the physical significance of 'V' Number of an optical fibre.

**2.** Answer ***any three*** questions :

(a) Write the necessary conditions for lasing action. Show that lasing action is not possible in two-level Laser system. 2+3

(b) Explain the basic principle of a Laser resonator. 5

(c) Explain Laser cooling and its application for ion trapping. 5

(d) What are the sources of attenuation in an optical fibre? If the output power is half of the input power, then find out the attenuation of the optical beam. 2+3

(e) Suppose the field incident on a non-linear dielectric medium is given by $E = E_0 \cos \omega t$. Calculate the polarization $P$ and identify the second harmonic term. Explain why the second harmonic term cannot occur in isotropic medium. 3+2

**3.** (a) What are the fundamental modes of vibration of $CO_2$ molecule?

(b) Explain the working principle of $CO_2$ laser with energy diagram and wavelength of laser radiation.

(c) Explain the role of Ne in He-Ne Laser with proper diagram. 2+5+3

**4.** (a) Write down the rate equations in four-level laser system and draw the energy level diagram indicating the metastable state and light emission.

(b) Why four-level laser is more efficient than three-level laser?

(c) What fraction of sodium atom to be in the first excited state in sodium vapour lamp at temperature of 523 K [$\lambda = 590$ nm]? (2+3)+2+3

**5.** (a) Explain Q-switching and mode-locking of laser line.

(b) For what purpose these techniques are used?

(c) Show that the output of a mode-locked laser is a series of pulses at a time interval of $\frac{2\mu L}{C}$, which is equal to the roundtrip transit time for light within the cavity of length $L$ and r.i. $\mu$ and each pulse has maximum intensity equal to $N$ times of average intensity. 3+2+5

**6.** (a) Explain with necessary diagram the working principle of Nd : YAG Laser.

(b) For a system in thermal equilibrium, calculate the temperature at which the spontaneous emission rate and stimulated emission rate are equal for a wavelength of 500 nm.

(c) Define the line shape function of Laser. What are the different types of line broadening mechanism? Explain Doppler broadening mechanism of Lasing system. 3+3+4

**7.** (a) For a step index fibre with core diameter $d$, show that the distance between two successive reflections is given by

$$L = d\sqrt{\left(\frac{\mu_1}{\mu_0 \sin\theta_i}\right)^2 - 1},$$

where, $\mu_1$ and $\mu_0$ are the refractive indices of core and surrounding medium respectively and $\theta_i$ is the angle of incidence.

(b) (i) Draw a generic block diagram of a fibre optical communication system. What is the speciality of guided mode?

(ii) What are the advantages of using optical fibre sensors? 5+{(2+1)+2}

**8.** (a) Distinguish between linear and non-linear optics.

(b) To observe non-linear effect what type of light source and media are required?

(c) Give examples of two non-linear crystals.

(d) A glass-clad fibre is made with core glass of refractive index 1.5 and the cladding is doped to give a fractional index difference of 0.0005.

Find :

(i) the cladding index

(ii) numerical aperture

(iii) the external critical acceptance angle. 2+2+1+(1+2+2)