Paper : DSE-B-2.2

(Advanced Statistical Mechanics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer ***any five*** questions : 2×5

(a) State the two postulates of quantum statistical mechanics.

(b) The mean occupation number of a single particle state for the Fermions and Bosons can be expressed as $n(\epsilon)=\dfrac{1}{e^{(\epsilon-\mu)/k_BT}\pm 1}$.

Plot the functional behaviour of $n(\epsilon)$ with $(\epsilon-\mu)/k_BT$ for both the cases.

(c) Consider the following density matrix :

$$\rho=\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}.$$

Does it represent a 'pure state' or a 'mixed state'? Justify your answer.

(d) Define Chandrasekhar limit.

(e) The Hamiltonian of a system is given as $H=Ap_x^{\,2}+Cx^2+Kx$, where A, C and K are constants. Determine $\langle H\rangle$.

(f) Show that the entropy is given as $S=\Sigma_i P_i \ln P_i$ [When P_i is the probability that any arbitrary system from the ensemble has an energy E_i].

(g) Write the partition function of a two Boson system having single particle energy levels ϵ, 2ϵ and 3ϵ.

Group - B

2. Answer ***any three*** questions :

(a) State Liouville's theorem for the phase space characterized by a probability density $\rho(q, p, t)$. Hence establish the condition required for a system to be in statistical equilibrium. 3+2

(b) Show that the mean number of Bosons in the ground state is (using grand canonical partition function)

$$n(\epsilon_0)=\frac{1}{e^{(\epsilon_0-\mu)/k_BT}-1}.$$

Hence, show that for all finite values of T, the chemical potential μ for the Bosons is always less than the ground state energy ϵ_0. 5

(c) The Hamiltonian for a single particle in an extreme relativistic ideal gas is $H = c\sqrt{p_x^2 + p_y^2 + p_z^2}$. Apply the generalized equipartition theorem to find $\langle H \rangle$. 5

(d) (i) Write the expression of grand canonical partition function of classical ideal gas explaining all the symbols.

(ii) Hence, determine the chemical potential μ for a classical ideal gas. 2+3

(e) Show that for a two-dimensional free electron gas, number of electron per unit area is

$$n = \frac{4\pi mkT}{h^2} \ln\left(e^{E_F/kT} + 1\right)$$ 5

Group - C

Answer *any four* questions.

3. (a) Consider a system composed on N non-interacting, distinguishable two-level atoms with energy $+\epsilon$ when they are up and $-\epsilon$ when they are down. Calculate the entropy of this system as a function of energy E. Hence, discuss under what condition can the absolute temperature be negative.

(b) A random walker in one-dimension takes steps to the left or right with equal probability. Calculate the probability that the walker starting from the origin is back to the origin after taking even number of steps N. What will be the probability if N is odd? (4+1)+(4+1)

4. (a) Given that the canonical partition function for classical ideal gas has an expression $Z(N,V,T) = \frac{1}{N!}\frac{V^N}{\lambda^{3N}}$, obtain the partition function for classical ideal gas in grand canonical ensemble. Hence, calculate the mean particle number $\langle N \rangle$ and the chemical potential of the system. Here $\lambda = h/\sqrt{2\pi mk_BT}$ is the mean thermal wavelength of the system.

(b) Show that the fluctuations in the number density (n) in grand canonical ensemble can be expressed as $\langle n^2 \rangle - \langle n \rangle^2 = \frac{k_BT}{V^2}\frac{\partial}{\partial\mu}\langle n \rangle$. Also find the relative root mean square fluctuation in n.

(3+2)+(4+1)

5. (a) Deduce an expression for Bose-Einstein distribution function from the grand partition function of an ideal Bose gas.

(b) What do you mean by Bose-Einstein condensation?

(c) Find the equation of state of an ideal Bose gas in the condensed phase. 3+2+5

6. (a) The Hamiltonian for the Ising model with N spins is

$$H = -J\sum_{\langle ij\rangle} s_i s_j ,$$

where $s_i = \pm 1$, $J > 0$ and $\Sigma_{\langle ij\rangle}$ is a sum over nearest neighbours.

Show that, within Bragg Williams approximation, the mean magnetization per spin can be expressed as $\bar{m} = \tanh\left(J\gamma\bar{m}/k_B T\right)$, where γ is the number of nearest neighbours. Study the solution graphically for $\bar{m}$ to show that there exists a critical temperature T_C below which the system can have a non-zero spontaneous magnetization and above cannot.

(b) As a simplified model of a binary alloy, consider a square lattice of atoms which can be either of type 1 or type 2. Set spin values $s_1 = +1$ and $s_2 = -1$ and let there be N_1 number of type 1 atoms and N_2 number of type 2 atoms, such that $N_1 + N_2 = N$. Let the interaction energies between two neighbouring atoms be ε_{11}, ε_{22} and ε_{12}, and there be N_{11}, N_{22} and N_{12} bonds of each type. The energy of the binary alloy is thus $E = \varepsilon_{11}N_{11} + \varepsilon_{22}N_{22} + \varepsilon_{12}N_{12}$.

Show that the above expression for E can be written in the following form :

$$E = -J\sum_{\langle ij\rangle} s_i s_j - h\sum_i s_i - cN$$

Find the expression for J, h, and c. 6+4

7. (a) Derive an expression for Fermi energy in terms of particle density.

(b) Show that the degeneracy pressure of a strongly degenerate Fermi gas varies with the number density as $P_0 \sim n^{5/3}$. 5+5

8. (a) For an isolated solid of N molecules, where the molecules are on a fixed 3-dimensional lattice and they vibrate like classical linear harmonic oscillators with a common frequency v. Determine the entropy S, the internal energy U and the Helmholtz free energy F in terms of N, v and temperature T. Could the system be treated as entirely a classical system? Justify your answer.

(b) For an isolated ideal gas derive the Sackur Tetrode equation. Is Boltzmann's counting truly justified in this case? (2+2+1+1)+4