

2024

MATHEMATICS — HONOURS

Paper : CC-8

(Riemann Integration and Series of Functions)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.* $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the sets of natural, real, rational numbers and integers respectively.

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify your choice. (1+1)×10

(a) Let $f(x)$ be a bounded real valued function defined on $[a, b]$ and P be a partition on $[a, b]$, then

(i) $L(P, -f) = -L(P, f)$

(ii) $L(P, -f) = U(P, f)$

(iii) $L(P, -f) = -U(P, f)$

(iv) $U(P, -f) = -U(P, f)$

(b) A continuous function $f: [-\pi, \pi] \rightarrow \mathbb{R}$,

(i) always has a primitive in $[-\pi, \pi]$ (ii) may not be Riemann integrable in $[-\pi, \pi]$

(iii) has a primitive only when it is derivable

(iv) is derivable if it has a primitive in $[-\pi, \pi]$.

(c) Let $f: [0, 2024] \rightarrow \mathbb{R}$ be defined by

$$f(x) = 1, \text{ if } x \in [0, 2024] \cap \mathbb{Z} \\ = 0, \text{ otherwise}$$

The value of $\int_0^{2024} f(x)dx$ is

(i) 2024

(ii) does not exist

(iii) 1

(iv) 0.

(d) Let f and g be two real valued functions defined on $[a, b]$ as follows :

$$f(x) = \begin{cases} 5, & x \in \mathbb{Q} \cap [a, b] \\ 1, & x \in [a, b] - \mathbb{Q} \end{cases}$$

$$g(x) = \begin{cases} -3, & x \in \mathbb{Q} \cap [a, b] \\ 1, & x \in [a, b] - \mathbb{Q} \end{cases}$$

Which of the following functions is Riemann integrable on $[a, b]$?

- (i) $f + g$ (ii) $f - g$
(iii) $f \cdot g$ (iv) f / g .

(e) If $B(k, 3) = \frac{1}{60}$ for the beta function B , the value of the positive integer k is

- (i) 2 (ii) 3
(iii) 4 (iv) 5.

(f) If $I_1 = \int_0^1 \frac{1}{x^\mu} dx$ and $I_2 = \int_0^1 \frac{1}{x^{2\mu}} dx$, where $\mu > 0$, then

- (i) I_2 exists finitely but not I_1 (ii) only I_1 exists finitely
(iii) if I_1 exists then I_2 exists (iv) if I_2 exists then I_1 exists.

(g) For any continuous function $f(x)$, the value of $\lim_{n \rightarrow \infty} \int_0^{\frac{1}{2}} x^n f(x) dx$ is

- (i) 1 (ii) 0
(iii) -1 (iv) $\frac{1}{2}$.

(h) The series $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$ is

- (i) uniformly convergent for all real x .
(ii) uniformly convergent only for positive real values of x .
(iii) convergent but not uniformly convergent on $\mathbb{R}$.
(iv) divergent everywhere.

(3)

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- (i) If R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$, then the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{a_n}{\log(2024)} x^n \text{ is}$$

(i) $\frac{R}{\log(2024)}$

(ii) R

(iii) $\frac{R}{2024}$

(iv) $R \log(2024)$.

- (j) The value of $\int_{-\infty}^{\infty} e^{-x^2} dx$ is

(i) $\sqrt{\pi}$

(ii) $\frac{\sqrt{\pi}}{2}$

(iii) $2\sqrt{\pi}$

(iv) $\sqrt{\frac{\pi}{2}}$.

2. Answer **any three** questions :

- (a) Let f be a real valued non-negative continuous function on $[a, b]$ with $\int_a^b f(x) dx = 0$. Show that

$f(x) = 0, \forall x \in [a, b]$. Also prove, by an example, that the continuity hypothesis cannot be dropped. 3+2

- (b) (i) Prove that $\frac{1}{4} \leq \int_0^{1/4} \frac{dx}{\sqrt{1-x^2}} \leq \frac{1}{\sqrt{15}}$

(ii) Evaluate $\lim_{x \rightarrow 0} \frac{\int_0^{x^4} \sin \sqrt{t} dt}{x^2}$

3+2

Please Turn Over

(c) If $f: [a, b] \rightarrow \mathbb{R}$ is a monotonically increasing function, then prove that for every $\varepsilon > 0$, there is $\delta > 0$ such that $U(P, f) - L(P, f) < \varepsilon$ for every partition P of $[a, b]$ with $\|P\| < \delta$. 5

(d) State and prove fundamental theorem of Integral Calculus. 1+4

(e) (i) Prove using Lebesgue's Criteria, that, if $f: [0, 1] \rightarrow \mathbb{R}$ and $g: [0, 1] \rightarrow \mathbb{R}$ are Riemann integrable in $[0, 1]$, then $fg: [0, 1] \rightarrow \mathbb{R}$ is also Riemann integrable in $[0, 1]$.

(ii) Use First Mean Value theorem of integral calculus to prove that $\frac{16\sqrt{2}}{41\pi} \leq \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{\sin \pi x}{2+x^2} dx \leq \frac{16\sqrt{2}}{33\pi}$.

2+3

3. Answer *any two* questions :

(a) (i) Examine the convergence of $\int_0^1 \frac{e^x}{\sqrt{1-\cos x}} dx$.

(ii) Examine the convergence of $\int_{-\infty}^{-1} \frac{e^x}{x^2} dx$.

3+2

(b) Show that the integral $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ converges if and only if $m, n > 0$.

5

(c) Prove that $\int_0^{\infty} \frac{\sin x}{x} dx$ is conditionally convergent.

5

(d) (i) If $0 < \mu < 1$, then prove that $\int_a^b (x-a)^{-\mu} dx$ converges.

(ii) Examine the convergence of $\int_0^{\frac{\pi}{2}} \log \sin x dx$.

2+3

4. Answer *any four* questions :

- (a) Prove that a sequence $\{f_n\}_{n=1}^{\infty}$ of real valued functions defined in $E \subseteq \mathbb{R}$ converges uniformly in E if and only if for every $\epsilon > 0$, there is $k \in \mathbb{N}$ such that $\forall m, n > k$ and $\forall x \in E$, $|f_n(x) - f_m(x)| < \epsilon$. 5

- (b) (i) Test the uniform convergence of the sequence of functions $\left\{ \frac{n^2 x}{1 + n^4 x^2} : x \in [0, 1] \right\}$.

- (ii) Give an example of sequence of functions that converges pointwisely on $[0, 1]$ but does not converge uniformly on $[0, 1]$. 3+2

- (c) Suppose $\{f_n\}_n$ be a sequence of functions in $E \subseteq \mathbb{R}$ such that for every $n \in \mathbb{N}$, there is $M_n > 0$

such that $|f_n(x)| \leq M_n \forall x \in E$. If $\sum_{n=1}^{\infty} M_n$ converges, prove that the function $\sum_{n=1}^{\infty} f_n$ converges

uniformly in $E \subseteq \mathbb{R}$. With suitable example, show that the converse of this result is not true. 3+2

- (d) Establish the convergence of the series $e^{-x} + 2e^{-2x} + 3e^{-3x} + \dots$, for all $x > 0$ and hence, find the sum function. 3+2

- (e) If a power series $\sum_{n=0}^{\infty} a_n x^n$ is neither nowhere convergent nor everywhere convergent, prove that

there exists a real number $R > 0$ such that the series converges absolutely in $(-R, R)$ and does not converge for all x satisfying $|x| > R$. 5

- (f) (i) Find the region of convergence of the power series $x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \dots$

- (ii) If the sum of the power series $x + \frac{x^2}{2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \dots$ is $s(x)$ in its region of convergence,

then show that $s'(x) = 1 + \log(1+x)$ for $-1 < x \leq 1$. 3+2

- (g) Prove that $x^2 = \frac{\pi^2}{3} - 4 \left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right) \forall x \in [-\pi, \pi]$. 5