

2024

MATHEMATICS — HONOURS

Paper : CC-9

(Partial Differential Equation and Multivariate Calculus - II)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.**All symbols have their usual meaning.*

Group - A

(Marks : 20)

1. Answer all questions with proper justification (*one mark* for correct answer and *one mark* for justification) : (1+1)×10

(a) Nature of the partial differential equation $\frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial x^2} + z^2 \frac{\partial z}{\partial x} = 0$, where z is a function of

x and y , is

(i) linear and second order

(ii) linear and first order

(iii) non-linear and first order

(iv) non-linear and second order.

(b) For any arbitrary function f , the equation $x + y + z = f(x^2 + y^2 + z^2)$ satisfy the partial differential equation

(i) $(y - z)p - (z - x)q = (x - y)$

(ii) $(z - x)p + (y - z)q = (x - y)$

(iii) $(y - z)p + (z - x)q = (x - y)$

(iv) $(z - x)p + (y - z)q = (x - y).$

(c) For the partial differential equation $z^2 = pqxy$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, the Charpit's auxiliary equations are

$$(i) \quad \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{2pqxy} = \frac{dx}{-qxy} = \frac{dy}{pxy}$$

$$(ii) \quad \frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{-pxy}$$

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$$(iii) \frac{dp}{-pqy+2pz} = \frac{dq}{-pqx+2qz} = \frac{dz}{2pqxy} = \frac{dx}{-qxy} = \frac{dy}{-pxy}$$

$$(iv) \frac{dp}{-pqy+2pz} = \frac{dq}{-pqx+2qz} = \frac{dz}{2pqxy} = \frac{dx}{qxy} = \frac{dy}{pxy}.$$

(d) The partial differential equation $y^3 u_{xx} - (x^2 - 1) u_{yy} = 0$ will be hyperbolic if

(i) $y < 0, -1 < x < 1$

(ii) $y = 0, x = -1 \text{ or } +1$

(iii) $y > 0, x > 1 \text{ or } x < -1$

(iv) None of these.

(e) The characteristic curves of the partial differential equation $y^2 u_{xx} - x^2 u_{yy} = 0$ are given by

(i) $x^2 + y^2 = c_1, x^2 - y^2 = c_2$

(ii) $x^2 + y^2 = c_1, y^2 - 2x^2 = c_2$

(iii) $x^2 + y^2 = c_1, x^2 - 2y^2 = c_2$

(iv) $x^2 + y^2 = c_1, 2x^2 - y^2 = c_2.$

(f) If $\phi_x = \psi_y$ and $\phi_y = -\psi_x$, then ϕ and ψ satisfy :

(i) $\nabla^2 \phi \neq 0, \nabla^2 \psi \neq 0$

(ii) $\nabla^2 \phi \neq 0, \nabla^2 \psi = 0$

(iii) $\nabla^2 \phi = 0, \nabla^2 \psi = 0$

(iv) $\nabla^2 \phi = 0, \nabla^2 \psi \neq 0.$

(g) If the order of integration in $\int_1^2 dx \int_x^2 g(x, y) dy$ is reversed, then it will take the form

(i) $\int_1^2 dy \int_1^{2-y} g(x, y) dx$

(ii) $\int_1^2 dy \int_1^y g(x, y) dx$

(iii) $\int_1^2 dy \int_y^1 g(x, y) dx$

(iv) $\int_1^2 dy \int_y^2 g(x, y) dx.$

(h) The value of line integral $\oint_{\Gamma} (-4y dx + 6x dy)$, where Γ is the circle defined by $\Gamma : x^2 + y^2 = 1$, is

(i) 10π

(ii) 5π

(iii) 15π

(iv) None of these.

(i) Value of the integral $\iint \cos(x+y) dx dy$ over the domain $x = 0, y = \pi, y = x$ is

(i) 1

(ii) -2

(iii) 2

(iv) None of these.

- (j) If $\vec{H} = \vec{\nabla} \times \vec{A}$ and $\hat{n}$ is the outer drawn unit normal to any closed surface S , then the value of $\iint_S (\vec{H} \cdot \hat{n}) dS$ is

- (i) 4 (ii) 2
(iii) 1 (iv) 0.

Group - B

(Marks : 21)

Answer *any three* questions.

2. (a) Find the partial differential equation of the family of right circular cones whose axes coincide with the z -axis.
(b) Solve the partial differential equation $(y+z)\frac{\partial z}{\partial x} + (z+x)\frac{\partial z}{\partial y} = x+y$ by Lagrange's method. 3+4
3. (a) Find the complete integral of $px + qy = pq$ using Charpit's method.
(b) Find the integral surface of the equation $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ passing through the parabola $y^2 = 4x, z = 1$. 4+3
4. Derive one-dimensional heat equation of the form $u_t = ku_{xx}$ by stating all the basic assumptions. 7
5. Reduce the equation $y^2 u_{xx} - x^2 u_{yy} = 0$ to its canonical form. 7
6. Obtain the d'Alembert solution of the following Cauchy problem for the one-dimensional wave equation :
 $u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, t > 0,$
 $u(x, 0) = f(x), \quad x \in \mathbb{R},$
 $u_t(x, 0) = g(x), \quad x \in \mathbb{R},$
 where f and g are given continuously differentiable functions. 7

Group - C

(Marks : 24)

Answer *any four* questions.

7. Using differentiation under the sign of integration, prove that

$$\int_0^{\pi/2} \log(a^2 \cos^2 \theta + b^2 \sin^2 \theta) d\theta = \pi \log\left(\frac{a+b}{2}\right),$$

where $a > 0, b > 0$.

8. Changing the order of integration, show that $\int_0^1 dx \int_x^{1/x} \frac{y dy}{(1+xy)^2 (1+y^2)} = \frac{\pi-1}{4}$. 6
9. Evaluate $\iint_R \sqrt{4a^2 - x^2 - y^2} dx dy$, where R is the upper half of the circle $x^2 + y^2 = 2ax$. 6
10. Find the value of the integral $\iiint_E \frac{dx dy dz}{x^2 + y^2 + (z-2)^2}$, where $E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. 6
11. Prove that the field of force $\vec{F} = (y^2 \cos x + z^3) \hat{i} + (2y \sin x - 4) \hat{j} + (3xz^2 + 2) \hat{k}$ is conservative. Find the scalar potential for $\vec{F}$. Also find the work done in moving a particle in this field from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$. 2+2+2
12. Verify Stoke's theorem for the function $\vec{F} = x\hat{i} + z^2\hat{j} + y^2\hat{k}$ over the plane surface $x + y + z = 1$ lying in the first octant. 6
13. Prove that the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$ is $\frac{2}{9}(3\pi - 4)a^3$. 6
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