

2025

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.* $\mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{Q}$ denote the set of all natural, real, complex and rational numbers respectively.*Notations and symbols have their usual meanings.*

1. Answer all the following multiple choice questions. For each question 1 mark for choosing correct option and 1 mark for justification : 2×10

- (a) If d_1, d_2 are two metrics on a set X , then

- (i) $\min \{d_1, d_2\}$ is a metric on X .
- (ii) $d_1 \cdot d_2$ is a metric on X .
- (iii) $\max \{d_1, d_2\}$ is a metric on X .
- (iv) d_1^2 and d_2^2 both are metrics on X .

- (b) Let A, B be subsets of a metric space (X, d) . Choose the false statement.

- (i) $\partial A \cap A^\circ = \emptyset$
- (ii) $\text{ext}(A \cup B) = \text{ext } A \cup \text{ext } B$
- (iii) $d(\bar{A}, \bar{B}) = d(A, B)$
- (iv) $\bar{A} = A \cup \partial A$.

- (c) Let (X, d_1) and (Y, d_2) be two metric spaces and $f: X \rightarrow Y$ be a continuous mapping. Which of the following is correct?

- (i) $f(\bar{A}) = \overline{f(A)}$, for all $A \subseteq X$
- (ii) $f(F)$ is closed in Y when $F \subseteq X$ is closed in X
- (iii) $f^{-1}(F)$ is closed in X when F is closed in Y
- (iv) $f^{-1}(\bar{B}) \subseteq \overline{f^{-1}(B)}$, for all $B \subseteq Y$.

- (d) Let X be an infinite set and $d: X \times X \rightarrow \mathbb{N} \cup \{0\}$ be a metric on X . Then every singleton set in (X, d) is

- (i) open but not necessarily closed.
- (ii) closed but not necessarily open.
- (iii) both open and closed.
- (iv) neither open nor closed.

Please Turn Over

(e) Which of the following functions is a contraction map on $[0, 1]$?

(i) $f(x) = x^2$

(ii) $f(x) = \sin x$

(iii) $f(x) = \frac{3x}{4}$

(iv) $f(x) = 2x - x^2$

(f) If L is the line segment from $(1, 0)$ to $(1, 1)$, then $\int_L \bar{z} dz$ is equal to

(i) $\frac{1}{2} - i$

(ii) $\frac{1}{2} + i$

(iii) i

(iv) $\frac{1}{2}$

(g) The bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$, $w_3 = -1$ is

(i) $w = \frac{5z+2}{3z+1}$

(ii) $w = \frac{3z+2}{iz+6}$

(iii) $w = \frac{iz+5}{2z+i}$

(iv) $w = \frac{-z+i}{3z-i}$

(h) By Cauchy's integral formula, the value of the integral $\int_C \frac{dz}{z(z+\pi i)}$, where C is $|z+3i|=1$, is

(i) 1

(ii) -1

(iii) 2

(iv) -2.

(i) The radius of convergence of the series, $\sum \left(1 + \frac{1}{n}\right)^{n^2} \cdot z^n$ is

(i) 1

(ii) ∞

(iii) e

(iv) $\frac{1}{e}$

(j) Let $f(z) = \bar{z}$, then which one of the following is true?

(i) $\frac{d}{dz}(f(z))$ exist only at $z=0$

(ii) $\frac{d}{dz}(f(z))$ exist everywhere except at $z=0$

(iii) $f(z)$ is analytic everywhere

(iv) $f(z)$ is non-analytic everywhere.

Unit - I

(Metric Space)

Answer *any five* questions.

2. Let l_2 denote the set of all real sequences $\{x_n\}$ for which $\sum_{n=1}^{\infty} |x_n|^2 < \infty$ and let

$$d(\{x_n\}, \{y_n\}) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^2 \right\}^{\frac{1}{2}}. \text{ Show that } (l_2, d) \text{ is a metric space.} \quad 5$$

3. (a) Prove that a subset A of a metric space (X, d) is open if and only if $A \cap \partial A = \emptyset$ (∂A = boundary of A).
 (b) Let (Y, d_Y) be a metric subspace of a metric space (X, d) . Prove that the interior of a set $A (\subseteq Y)$ in Y contains A° , the interior of A in (X, d) . 3+2
4. Let (X, d_1) and (Y, d_2) be metric spaces. Show that $f: X \rightarrow Y$ is continuous if and only if $f(\bar{A}) \subseteq \overline{f(A)}$ for every $A \subseteq X$. 5

5. (a) Show that, for any subset A of a metric space (X, d) , the function $f: X \rightarrow \mathbb{R}$ given by $f(x) = d(x, A)$, is uniformly continuous.
 (b) Let (X, d_1) and (Y, d_2) be metric spaces and $f: (X, d_1) \rightarrow (Y, d_2)$ be uniformly continuous. Show that, if $\{x_n\}$ is a Cauchy sequence in (X, d_1) then so is $\{f(x_n)\}$ in (Y, d_2) . 2+3
6. (a) Prove that every connected subset (with at least two points) in a metric space is uncountable.
 (b) If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(\mathbb{Q}) \subseteq \mathbb{R} - \mathbb{Q}$ and $f(\mathbb{R} - \mathbb{Q}) \subseteq \mathbb{Q}$, then prove that f cannot be continuous. 3+2

7. (a) Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X . Then show that $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

- (b) Show that in a metric space any open ball is an open set. 3+2

8. (a) Prove that any continuous function on a compact metric space (X, d_1) to a metric space (Y, d_2) is uniformly continuous.
 (b) Let (X, d) be a compact metric space and let A be a subset of X . Show that A is compact if it is closed. 3+2

Please Turn Over

9. Let (X, d) be a complete metric space and let f be a contraction mapping on X . Prove that there exists one and only one point x in X such that $f(x) = x$. 5

Unit - II

(Complex Analysis)

Answer *any four* questions.

10. (a) Find the stereographic projection of the point $z = x + iy$ on the Riemann sphere $S: x_1^2 + x_2^2 + x_3^2 = 1$.

(b) Find $\lim_{z \rightarrow \frac{i\pi}{3}} \frac{z^3 + 1}{z^4 + z^2 + 1}$. 3+2

11. (a) Let $f: G \rightarrow \mathbb{C}$, where G is a region. Suppose the sequence $\{f(z_n)\}$ converges to L for every sequence of points $\{z_n\}$ in G , $z_n \neq z_0$ ($n = 1, 2, \dots$) and $z_n \rightarrow z_0$ as $n \rightarrow \infty$, then prove that

$$\lim_{z \rightarrow z_0} f(z) = L.$$

(b) Let $f(Z) = \begin{cases} \frac{(\operatorname{Re} Z)(\operatorname{Im} Z)}{|Z|^2}, & Z \neq 0 \\ 0, & Z = 0. \end{cases}$

Is the function $f(Z)$ continuous at $Z = 0$? Justify.

3+2

12. (a) Let G be a non-empty open connected subset of $\mathbb{C}$ and $f: G \rightarrow \mathbb{C}$ be a function.

If $f'(z) = 0$ in G , prove that f is constant in G .

- (b) Prove that the function $f: \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) = x^2 + y^2 + ixy$; $z = x + iy$ is nowhere analytic. 3+2

13. Let $f: G \rightarrow \mathbb{C}$, where $f(x + iy) = u(x, y) + iv(x, y)$ be a function of a complex variable on a region G . Let $u(x, y)$, $v(x, y)$ be differentiable at (x_0, y_0) and let the Cauchy Riemann equations be satisfied at (x_0, y_0) , then prove that f is differentiable at $z_0 = x_0 + iy_0$. 5

14. (a) Prove that the transformation, $w = \frac{i(z-i)}{z+i}$ maps the upper half of z -plane into the interior of the unit circle in the w -plane.

- (b) Find the bilinear transformation which transforms $z_1 = 2$, $z_2 = 1$, $z_3 = 0$ into $w_1 = 1$, $w_2 = 0$ and $w_3 = i$. 2+3

15. (a) State Cauchy's Integral formula. Using it calculate the following integral :

$$\int_C \frac{z \, dz}{(9 - z^2)(z + i)},$$

where C is the circle $|z| = 2$ described in positive sense.

- (b) Evaluate $\int_C \frac{z^2 - 4}{z(z^2 + 9)} dz$, where C is the circle $|z| = 1$. (1+2)+2

16. (a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ be the power series obtained by differentiating the series $\sum a_n z^n$ term by term. Then the derived series has the same radius of convergence as the original series.

- (b) Find the radius of convergence of the series $\sum \frac{2^{-n}}{1 + in^2} z^n$. 3+2
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