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Quantum Mechanics and Applications

Generalized Angular Momenta and Spin

Angular Momentum:

Here in this section we devote to the study of angular momenta in quantum mechanics. This is an extremely important problem, and the results are established in may domains of physics: such as the classification of atomic, molecular and nuclear spectra, the spin of elementary particles, origin of magnetism etc.
We already know the important role played by angular momentum in classical mechanics: the total angular momentum of an isolated physical system is a constant of the motion. This as also true in certain cases where the system is not isolated. For example if a particle P, of mass m, is moving in a central potential, the force to which P is subjected is always directed towards a fixed point O. Its moment with respect to O is consequently zero and the angular momentum implies that,

$$\frac{d\mathscr{L}}{dt} = 0 \tag{1}$$

where $\mathscr{L}$ is the angular momentum of P with respect to O. This fact has important consequences: the motion of the particle P is limited to a fixed plane; moreover this motion obeys the law of constant areal velocity (Kepler's 2nd Law). All these properties have their equivalents in quantum mechanics. With the angular momentum $\mathscr{L}$ of a classical system is associated an observable L, actually a set of observables, L_x, L_y and L_z which corresponds to the three components of L in a Cartesian frame. If the physical system under study is a point moving in a central potential, L_x, L_y and L_z are constants in quantum mechanical sense, i.e. they commute with the Hamiltonian H, describing the particle in the central potential $V(r)$. This important property considerably simplifies the determination and classification of eigenstates of H.

Orbital Magnetic Dipole Moment:

Consider an electron of mass m and charge $-e$ moving with velocity of magnitude v in a circular Bohr orbit of radius r (shown in the following figure 1). The charge circulating in a loop constitutes a current of magnitude,

$$i = \frac{e}{T} = \frac{ev}{2\pi r} \tag{2}$$

where, T is the orbital period and whose charge has magnitude e. In elementary electromagnetic theory, it is shown that such a current loop produces a magnetic field which is the same at large distances from the loop as that of a magnetic dipole located at the center of the loop and oriented perpendicular to its plane. For a current i in a loop of area A, the magnitude of orbital magnetic dipole moment μ_l of the equivalent dipole is

$$\mu_l = iA \tag{3}$$

and the direction of the magnetic dipole moment is perpendicular to the plane of the orbit. The following figure shows the magnetic field produced by the magnetic field.

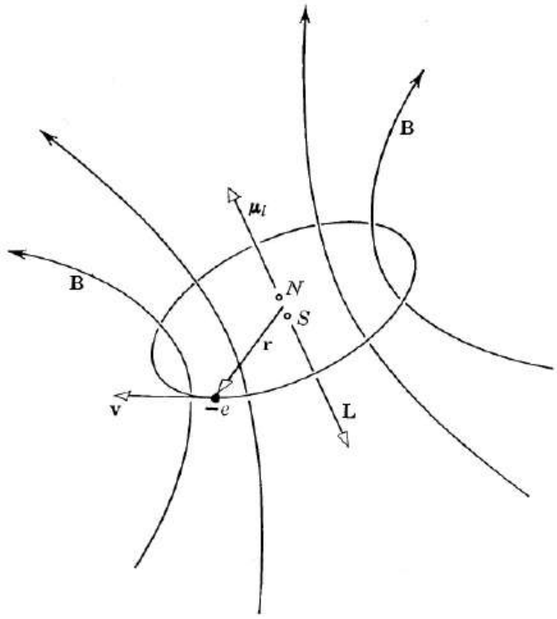

fig1

Its also indicates the two fictitious poles of a dipole that would produce a magnetic field which becomes identical to he actual field far from the loop. The quantity μ_l specifies the strength of this magnetic dipole; it equals the product of the poles' strength times their separation. Because the electron has a negative charge, its magnetic dipole moment μ_l is antiparallel to its orbital angular momentum L, whose magnitude is given by,

$$L = mvr \tag{4}$$

and whose direction is shown in the above figure. Now evaluating i from equation (2), and A for a circular orbit, yields

$$\mu_l = iA = \frac{ev}{2\pi r}\pi r^2 = \frac{evr}{2} \tag{5}$$

Now dividing equation (5) by equation (4), we therefore get,

$$\frac{\mu_l}{L} = \frac{evr}{2mvr} = \frac{e}{2m} \tag{6}$$

We see the ratio of the magnitude μ_l of the orbital dipole moment to the magnitude L of the orbital angular momentum for the electron is a combination of Universal constant. It is usual to write the ratio as,

$$\frac{\mu_l}{L} = \frac{g_l \mu_b}{\hbar} \tag{7}$$

where,

$$\mu_b = \frac{e\hbar}{2m} = 0.927 \times 10^{-23} amp - m^2 \tag{8}$$

and

$$g_l = 1 \tag{9}$$

The quantity μ_b forms a natural unit for the measurement of atomic magnetic dipole moments, and is called the Bohr magneton. The quantity g_l is called the orbital g factor.

Vector Atomic Model:

The limitations of old quantum theory compels the then Scientists introduce some idea partly by analogy and partly by empirically which finally resulted in which is known as Vector Atomic Model (V.A.M.). Two essential elements that characteristics the V.A.M. are-

(i) The concept of space quantization

(ii) Spin electron hypothesis

Space Quantization

The Bohr-Sommerfeld orbits are quantized only in the magnitude but in V.A.M. it's orientation is also quantized. It selects from a continuous manifold of all possible orientations permitted in classical physics, a discrete number conformable to the quantum condition. For example an orbit with orbital quantum number l may take their orientation $(2l + 1)$ in space.

To quantized specially we need of course a certain preferred direction with respect to which the orbits may receive their orientation. A magnetic field applied on the atom is considered as the reference direction. To determine the permitted orientations relative to the field direction we are guided by the fact that the projections of the quantized orbits on the field direction must themselves be quantized.

Thus if l is the orbital quantum number of the orbit then its orbital angular momentum will have values $l(l + 1)$ and in a magnetic field H it takes $(2l + 1)$ orientations, where the projections of the orbital angular momentum on the H will have the values

$$p_l = \sqrt{l(l+1)}\hbar \tag{10}$$

Spin Electron Hypothesis

In order to explain some of he intricate spectral phenomena such as fine structure of Sodium (Na) D-line, anomalous Zeeman effect etc, two Dutch Physicists Uhlenbeck ad Goudsmit in 1926 put forward the hypothesis of spining electron. According to them an electron rotates around themselves which will produce an angular momentum and also magnetic moment to the charged particle. Thus an electron in an atom will have two angular momentum-

(1) Orbital angular momentum

$$\sqrt{l(l+1)}\hbar$$

(2) Spin angular momentum

$$\sqrt{s(s+1)}\hbar$$

and hence we get two magnetic moments; one is the orbital magnetic moment and other is the spin magnetic moment.

Stern-Gerlach Experiment:

This experiment, first performed in 1922, has long been considered as the quintessential experiment that illustrates the fact that the electron possesses intrinsic angular momentum, i.e. spin. It is actually the case that the original experiment had nothing to do with the discovery that the electron possessed spin: the first proposal concerning the spin of the electron, made in 1925 by Uhlenbach and Goudsmit, was based on the analysis of atomic spectra. What the experiment was intended to test was 'space-quantization' associated with the orbital angular momentum of atomic electrons. The prediction, already made by the 'old' quantum theory that developed out of Bohr's work, was that the spatial components of angular momentum could only take discrete values, so that the direction of the angular momentum vector was restricted to only a limited number of possibilities, and this could be tested by making use of the fact that an orbiting electron will give rise to a magnetic moment proportional to the orbital angular momentum of the electron. So, by measuring the magnetic moment of an atom, it should be possible to determine whether or not space quantization existed. In fact, the results of the experiment were in agreement with the then existing (incorrect) quantum theory − the existence of electron spin was not at that time suspected. Later, it was realized that the interpretation of the results of the experiment were incorrect, and that what was seen in the experiment was direct evidence that electrons possess spin. It is in this way that the Stern-Gerlach experiment has subsequently been used, i.e. to illustrate the fact that electrons have spin. But it is also valuable in another way. The simplicity of the results of the experiment (only two possible outcomes), and the fact that the experiment produces results that are directly evidence of the laws of quantum mechanics in action makes it an ideal means by which the essential features of quantum mechanics can be seen and, perhaps, 'understood'.
First direct and convincing confirmation of the essential features of the V.A.M. was given by Stern and Gerlach. In 1922 Stern and Gerlach measured the possible values of the magnetic dipole moment for silver atoms by sending a beam of these atoms through a non uniform magnetic field. A schematic diagram of of their apparatus is shown the figure below.

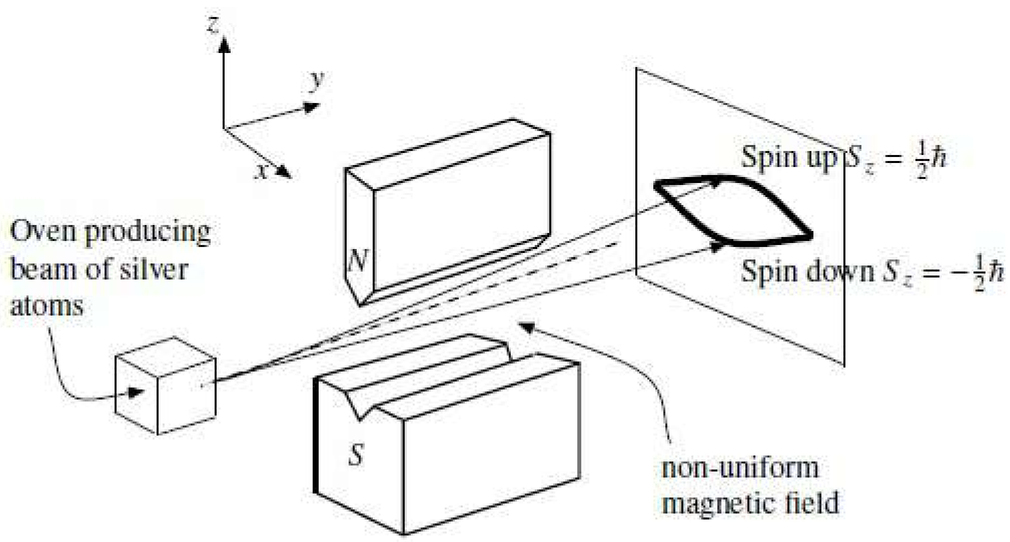

fig2

Principle

The atoms with it's magnetic moment arising out from the orbital and spin motion of the electrons in it may be regarded as an elementary magnet. If this atomic magnet is placed in a non-homogeneous magnetic field, the force on the two poles will not be equal and hence it is deviate away from its rectilinear path.

Theory

Let there is a magnetic field which is non-homogeneous along z-direction. Let the gradient of the magnetic field is $(\frac{dB}{dz})$. If an atomic magnet makes an angle θ with the direction of the magnetic field at any instant, then we can calculate the forces acting on the two poles in the following way. Suppose above the equal and opposite force m (where m be the pole strength) constituting a rotating couple, there is an extra force equal to

$$mlcos\theta\frac{dB}{dz} = \mu cos\theta\frac{dB}{dz} \tag{11}$$

Due to this fore the atom will be displaced along the direction of the magnetic field and the displacement of the atom will be given by,

$$\Delta = 0 + \frac{1}{2}\frac{\mu_z}{M_a}\frac{dB}{dz}(\frac{L}{v})^2 \tag{12}$$

where, M_a =mass of the atom, L =extension of the magnet, v =initial velocity of the electrons; $\frac{1}{2}M_a v^2 = \frac{3}{2}kT$

Apparatus

The schematic diagram of the apparatus used by Stern and Gerlach are shown above. Silver (Ag) atoms coming from the oven after being collimated by the slits and enters the space between the two magnetic poles and then taken on the photograph or screen. To obtain a large in-homogeneity in the magnetic field Stern and Gerlach constructed the pole pieces such that one pole is made of knife edge and the other is provided with a groove. Due to this shape the magnetic lines of force will crowded together at the knife edge so that the field strength is considerably greater.

Results and Discussions

The trace of the Ag-atom beam in the absence and the presence of the magnetic field are as show in the above figure. We see that in the presence of the magnetic field the atomic beam splits into two parts. We know the ground state of the Ag-atom is $2_{S_{\frac{1}{2}}}$, therefore according to V.A.M. the magnetic moment associated with the Ag-atom will have two orientations in the magnetic field, which is clearly obtained in the experiment. Stern-Gerlach measured the maximum deviation Δ and found that it was an excellent agreement with that calculated from equation(12).

Spin Angular Momentum:

To account for the multiplicity of atomic states, Uhlenbeck and Goudsmit proposed in 1925 that an electron in an atom possesses an intrinsic angular momentum in addition to orbital angular momentum. This intrinsic angular momentum S is called spin angular momentum whose projection on the z-axis can have the values,

$$S_z = m_s\hbar \tag{13}$$

where, $m_s = \pm\frac{1}{2}$. The maximum measurable component of spin angular momentum in units of $\hbar$ is called spin of the particle and is usually denoted by s. They also suggested that the spin angular momentum gives rise to an intrinsic magnetic moment μ_s,

$$\mu_s = -\frac{e}{m}S \tag{14}$$

Assuming that all the stable and unstable particles to have spin angular momentum S, we expect its components, S_x, S_y, and S_z to obey the general commutation relations:

$$[S_x, S_y] = i\hbar S_z, [S_y, S_z] = i\hbar S_x, [S_z, S_x] = i\hbar S_y \tag{15}$$

S^2 and S_z to have the eigenvalues $s(s+1)\hbar^2$ and $m_s\hbar$, $m_s = -s, -s+1, ..., s$ respectively.

Larmor's Theorem:

Let us consider a system, consisting a magnetic dipole moment μ_l in a magnetic field $\vec{B}$, to dissipate energy the orientational potential energy, ΔE of the system must remain constant. In these circumstances μ_l cannot align itself with $\vec{B}$. Instead μ_l will precess around $\vec{B}$ in such a way that the angle between these two vectors remain constant, and magnitudes of both vectors remain constant. In elementary electromagnetic theory it is shown that the dipole will experience a torque in a field $\vec{B}$,

$$\vec{\tau} = \mu_l \times \vec{B} \tag{16}$$

tending to align the dipole with the field. The precessional motion of that system due to the said fact, the torque acting on the dipole is always perpendicular to its angular momentum, in complete analogy to the case of a spinning top. The precession and its explanation are shown in figure below. It is clear to understand that the magnitude of the angular frequency of precession of μ_l about $\vec{B}$ is given by,

$$\vec{\omega} = \frac{g_l \mu_b}{\hbar}\vec{B} \tag{17}$$

This equation also indicates that the sense of the precession is in the direction of $\vec{B}$. The phenomenon is known as the Larmor precession, and ω is called the Larmor frequency.

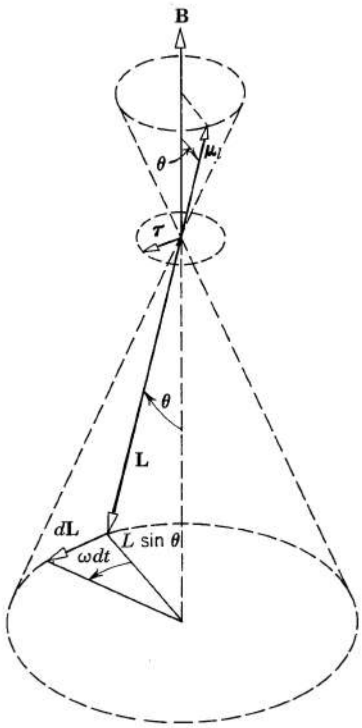

fig3

The g-factor:

We know that the orbital angular momentum of an atom with orbital quantum number l, is

$$G_L = \sqrt{l(l+1)}\hbar \tag{18}$$

and the spin angular momentum with the spin quantum number s, is

$$G_S = \sqrt{s(s+1)}\hbar \tag{19}$$

Therefore the magnetic moment associated with the orbital and spin motion are-

$$\mu_L = \frac{e}{2m} G_L \tag{20}$$

$$\mu_S = \frac{e}{m} G_S \tag{21}$$

Hence their z-components are-

$$\mu_{L_z} = \frac{e}{2m} l\hbar = \frac{e}{2m} G_{L_z} \tag{22}$$

$$\mu_{S_z} = \frac{e}{m} s\hbar = \frac{e}{m} G_{S_z} \tag{23}$$

Thus the total magnetic moment along z-axis will be,

$$\mu_z = \mu_{L_z} + \mu_{S_z} \tag{24}$$

Therefore,

$$\mu_z = \frac{e\hbar}{2m}(l + 2s) \tag{25}$$

Now it can be shown that the total z-component of the magnetic moment of an atom can be written as $J_z\beta g$, where J_z is the total angular momentum quantum number and g is a factor which is known as Lande-g-factor or the g-factor and β is known as Bohr magneton.

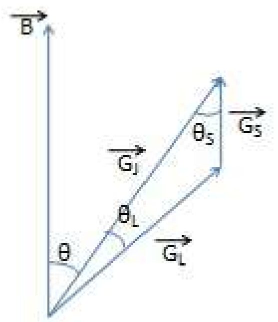

fig4

Let, $\vec{G_L}$, $\vec{G_S}$, $\vec{G_J}$ are the orbital, spin and total angular momentum of an atom whose magnitude are given by,

$$G_J^2 = J(J+1)\hbar^2 \tag{26}$$

$$G_L^2 = L(L+1)\hbar^2 \tag{27}$$

$$G_S^2 = S(S+1)\hbar^2 \tag{28}$$

Therefore, the component of the orbital, spin and total angular momentum along the direction of the external magnetic field are as follows-

$$G_{L_z} = G_L \cos\theta_L \cos\theta \tag{29}$$

$$G_{S_z} = G_S \cos\theta_S \cos\theta \tag{30}$$

$$G_{J_z} = G_J \cos\theta \tag{31}$$

Now the magnetic moment will have the z-component,

$$\mu_z = \frac{e}{2m}(G_{L_z} + 2G_{S_z}) \tag{32}$$

Therefore,

$$\mu_z = \frac{e}{2m}(G_L \cos\theta_L \cos\theta + 2G_S \cos\theta_S \cos\theta) \tag{33}$$

Now from the figure we get,

$$G_L^2 = G_J^2 + G_S^2 - 2G_J G_S \cos\theta_S \tag{34}$$

$$G_S^2 = G_J^2 + G_L^2 - 2G_J G_L \cos\theta_L \tag{35}$$

Therefore we get,

$$G_S \cos\theta_S = \frac{G_J^2 + G_S^2 - G_L^2}{2G_J} \tag{36}$$

and

$$G_L \cos\theta_L = \frac{G_J^2 + G_L^2 - G_S^2}{2G_J} \tag{37}$$

Therefore,

$$\mu_z = \frac{e}{2m} G_J \cos\theta [\frac{G_J^2 + G_L^2 - G_S^2}{2G_J^2} + \frac{G_J^2 + G_S^2 - G_L^2}{G_J^2}] = \frac{e}{2m} G_{J_z} [1 + \frac{1}{2} + \frac{G_L^2 - G_S^2 + 2G_S^2 - 2G_L^2}{2G_J^2}] \tag{38}$$

$$\mu_z = \frac{e}{2m} G_{J_z} [1 + \frac{G_J^2 + G_S^2 - G_L^2}{2G_J^2}] \tag{39}$$

Therefore we can write,

$$\mu_z = \frac{e}{2m} \hbar J_z [1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}] \tag{40}$$

Therefore finally we get,

$$\mu_z = J_z \beta g \tag{41}$$

where, $J_z = -J \rightarrow +J$ differing by an integer and

$$g = [1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}]$$

is called the g-factor.