Online Course Materials :

Prepared by: Masiur Rahaman Sardar

Assistant Professor

Dept. Of Mathematics

City College

Mobile : 9830458374

Email - sardarmasiur@gmail.com

Subject : Mathematics

Year/Semester : 1^{st} semester

Paper : CC-1 (Theory)

Unit/Chapter /Module : unit -3

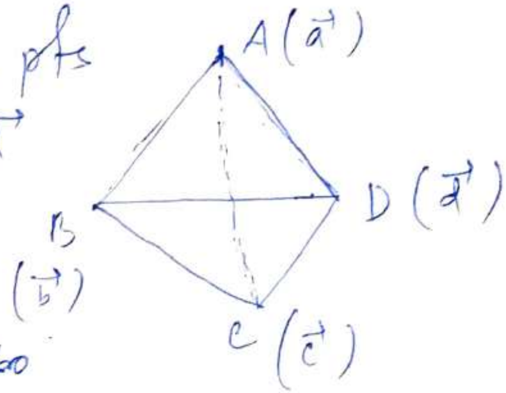
Topic / Title : Problems and solutions on product of three or more vectors

Page- 24/7/2017 Sub: Vector triple Product
QUES/PROB:

Show that the perpendicular distance of a pt whose position vector is $\vec{a}$ from the plane through three pts with position vector $\vec{b}, \vec{c}, \vec{d}$ is

$$\frac{[\vec{b}\,\vec{c}\,\vec{d}] + [\vec{c}\,\vec{a}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{b}\,\vec{a}\,\vec{c}]}{|\vec{b}\times\vec{c} + \vec{c}\times\vec{d} + \vec{d}\times\vec{b}|}$$

Solution: Let A, B, C, D be four pts whose p.vs are $\vec{a}, \vec{b}, \vec{c}$ & $\vec{d}$ respectively w.r.t a base pt not shown in the figure



Since B, C, D lie on a plane ~~and so~~ and A is outside the plane, so ABCD form a tetrahedron.

Let p be the perpendicular distance from the point A to the plane BCD.

Then volume of the tetrahedron ABCD $= \frac{1}{3}$ area of $\Delta BCD \times p$ ...(1)

Now, volume of the tetrahedron ABCD $= \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$$= \frac{1}{6}\left[(\vec{b}-\vec{a})\,(\vec{c}-\vec{a})\,(\vec{d}-\vec{a})\right]$$

$$= \frac{1}{6}\left\{(\vec{b}-\vec{a}).\left\{(\vec{c}-\vec{a})\times(\vec{d}-\vec{a})\right\}\right\}$$

$$= \frac{1}{6}\left\{(\vec{b}-\vec{a}).(\vec{c}\times\vec{d} - \vec{c}\times\vec{a} - \vec{a}\times\vec{d})\right\}$$

$$= \frac{1}{6}\left\{[\vec{b}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{a}] - [\vec{b}\,\vec{a}\,\vec{d}] - [\vec{a}\,\vec{c}\,\vec{d}]\right\}$$

$$= \frac{1}{6}\left\{[\vec{b}\,\vec{c}\,\vec{d}] + [\vec{c}\,\vec{a}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{b}\,\vec{a}\,\vec{c}]\right\}$$

Area of $\Delta BCD = \frac{1}{2} \left| \vec{BC} \times \vec{BD} \right|$

$$= \frac{1}{2} \left| (\vec{c} - \vec{b}) \times (\vec{d} - \vec{b}) \right|$$

$$= \frac{1}{2} \left| \vec{c} \times \vec{d} - \vec{c} \times \vec{b} - \vec{b} \times \vec{d} \right|$$

$$= \frac{1}{2} \left| \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b} \right|$$

Then from (1) we get

$$\frac{1}{6} \left\{ [\vec{b}\,\vec{c}\,\vec{d}] + [\vec{c}\,\vec{a}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{b}\,\vec{a}\,\vec{c}] \right\} = \frac{1}{3} \cdot \frac{1}{2} \left| \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b} \right| p$$

$$\text{or } p = \frac{[\vec{b}\,\vec{c}\,\vec{d}] + [\vec{c}\,\vec{a}\,\vec{d}] + [\vec{a}\,\vec{b}\,\vec{d}] + [\vec{b}\,\vec{a}\,\vec{c}]}{\left| \vec{b} \times \vec{c} + \vec{c} \times \vec{d} + \vec{d} \times \vec{b} \right|}$$

Which is the required result.

12(i) Show that $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ & $\vec{c}\times\vec{a}$ coplanar iff $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Solution: $[(\vec{a}\times\vec{b})\ (\vec{b}\times\vec{c})\ (\vec{c}\times\vec{a})]$

$$= (\vec{a}\times\vec{b})\cdot\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$$

$$= (\vec{a}\times\vec{b})\cdot\{((\vec{b}\times\vec{c})\cdot\vec{a})\vec{c} - ((\vec{b}\times\vec{c})\cdot\vec{c})\vec{a}\}$$

$$= (\vec{a}\times\vec{b})\cdot\{[\vec{a}\,\vec{b}\,\vec{c}]\vec{c} - [\vec{c}\,\vec{b}\,\vec{c}]\vec{a}\}$$

$$= (\vec{a}\times\vec{b})\cdot([\vec{a}\,\vec{b}\,\vec{c}]\vec{c})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}]((\vec{a}\times\vec{b})\cdot\vec{c})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}][\vec{c}\,\vec{a}\,\vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 \quad \cdots(1)$$

Let $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ & $\vec{c}\times\vec{a}$ be coplanar, then

$[(\vec{a}\times\vec{b})(\vec{b}\times\vec{c})(\vec{c}\times\vec{a})] = 0$. Then from (1),

$[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$\Rightarrow$ $\vec{a}$, $\vec{b}$ & $\vec{c}$ are coplanar

Again let $\vec{a}$, $\vec{b}$ & $\vec{c}$ be coplanar. Then

$[\vec{a}\,\vec{b}\,\vec{c}] = 0$ and hence from (1),

$[(\vec{a}\times\vec{b})(\vec{b}\times\vec{c})(\vec{c}\times\vec{a})] = 0$

$\Rightarrow$ $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ & $\vec{c}\times\vec{a}$ are coplanar

12(a) Show that the pts $(2,4,6)$, $(3,4,5)$, $(4,4,4)$ and $(5,4,3)$ are coplanar

(b). Prove that $[\vec{\beta}\ \vec{\gamma}\ \vec{\delta}]\vec{\alpha} - [\vec{\alpha}\ \vec{\gamma}\ \vec{\delta}]\vec{\beta} + [\vec{\alpha}\ \vec{\beta}\ \vec{\delta}]\vec{\gamma} - [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}]\vec{\delta} = \vec{0}$

Solution: Let A, B, C & D denote the pts $(2,4,6)$, $(3,4,5)$, $(4,4,4)$ and $(5,4,3)$ respectively.

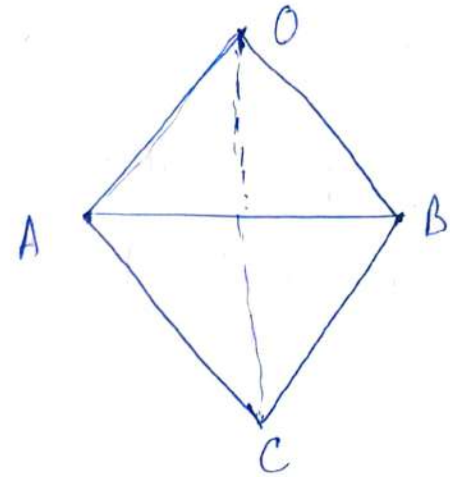
Then
$$\vec{AB} = (1, 0, -1)$$
$$\vec{AC} = (2, 0, -2)$$
$$\vec{AD} = (3, 0, -3)$$

Now,
$$[\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 3 & 0 & -3 \end{vmatrix} = 0$$

$\therefore$ $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar vectors and hence the pts A, B, C & D are coplanar

(b) C.H - 15 (same).

(15) Show that the volume of the tetrahedron whose vertices are given by the vectors $(-\hat{i}+\hat{j}+\hat{k})$, $(\hat{i}-\hat{j}+\hat{k})$, $(\hat{i}+\hat{j}-\hat{k})$ with reference to the fourth vertex as origin is $2/3$ cubic unit.



Solution: Let O be the origin. Then

$$\vec{OA} = -\hat{i}+\hat{j}+\hat{k}$$
$$\vec{OB} = \hat{i}-\hat{j}+\hat{k}$$
$$\vec{OC} = \hat{i}+\hat{j}-\hat{k}$$

We know that volume of the tetrahedron OABC is $\frac{1}{6}\left[\vec{OA}\ \vec{OB}\ \vec{OC}\right]$

$$= \frac{1}{6}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{1}{6}\left\{(-1)(0) - 1(-2) + 1(2)\right\}$$

$$= \frac{1}{6}(4) = \frac{2}{3}$$ cubic unit

13 (c) If $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ be three unit vectors such that $\vec{\alpha}\cdot\vec{\beta} = \vec{\alpha}\cdot\vec{\gamma} = 0$ and the angle between $\vec{\beta}$ and $\vec{\gamma}$ be 30°. Then show that $\vec{\alpha} = \pm 2(\vec{\beta}\times\vec{\gamma})$

Solution: Since $\vec{\alpha}\cdot\vec{\beta} = 0$ & $\vec{\alpha}\cdot\vec{\gamma} = 0$, so $\vec{\alpha}$ is perpendicular to both $\vec{\beta}$ and $\vec{\gamma}$ as all vectors are non zero.

Then $\vec{\alpha}$ and $\vec{\beta}\times\vec{\gamma}$ are parallel vectors

$$\therefore \vec{\alpha} = \lambda(\vec{\beta}\times\vec{\gamma}) \;\cdots\cdots (1)$$

$$\Rightarrow |\vec{\alpha}| = |\lambda|\,|\vec{\beta}\times\vec{\gamma}|$$

$$\Rightarrow 1 = |\lambda|\left\{|\vec{\beta}||\vec{\gamma}|\sin 30^\circ\right\}$$

$$\Rightarrow 1 = |\lambda|\frac{1}{2} \Rightarrow \lambda = \pm 2$$

Then from (1), $\vec{\alpha} = \pm 2 (\vec{\beta} \times \vec{\gamma})$.

(26) If $\hat{e}_1$ & $\hat{e}_2$ be two unit vectors and θ be the angle between them, then show that $2\sin\frac{1}{2}\theta = |e_1 - e_2|$

Solution: $|\hat{e}_1 - \hat{e}_2|^2 = (\hat{e}_1 - \hat{e}_2)\cdot(\hat{e}_1 - \hat{e}_2)$

$$= |\hat{e}_1|^2 - 2\hat{e}_1\cdot\hat{e}_2 + |\hat{e}_2|^2$$

$$= 2 - 2|\hat{e}_1||\hat{e}_2|\cos\theta$$

$$= 2 - 2\cos\theta$$

$$= 2(1-\cos\theta)$$

$$= 4\sin^2\theta/2$$

$\therefore |\hat{e}_1 - \hat{e}_2| = 2\sin\theta/2$ (as $|\vec{a}| \geq 0$ positive)

12(b) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ be three non-coplanar, non zero vectors, then show that any vector $\vec{d}$ can be expressed as

$$\vec{d} = \frac{[\vec{b}\,\vec{c}\,\vec{d}]\,\vec{a} + [\vec{c}\,\vec{a}\,\vec{d}]\,\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Solutions: Since $\vec{a}$, $\vec{b}$ & $\vec{c}$ are three non zeros non coplanar vectors, so $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

Since $\vec{a}$, $\vec{b}$ & $\vec{c}$ are non coplanar vectors, so we can express $\vec{d}$ as

$$\vec{d} = l\,\vec{a} + m\,\vec{b} + n\,\vec{c} \quad \ldots (1)$$ where

l, m & n are scalars to be determined.

Taking dot product from both sides of (1) by $\vec{b} \times \vec{c}$

we get

$$\vec{d}\cdot(\vec{b}\times\vec{c}) = l\,\vec{a}\cdot(\vec{b}\times\vec{c}) + m\,\vec{b}\cdot(\vec{b}\times\vec{c}) + n\,\vec{c}\cdot(\vec{b}\times\vec{c})$$

$$\Rightarrow [\vec{d}\,\vec{b}\,\vec{c}] = l[\vec{a}\,\vec{b}\,\vec{c}] + m[\vec{b}\,\vec{b}\,\vec{c}] + n[\vec{c}\,\vec{b}\,\vec{c}]$$

$$\Rightarrow l = \frac{[\vec{d}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} \quad \left(\because [\vec{x}\,\vec{y}\,\vec{x}] = 0\right)$$

$$= \frac{[\vec{b}\,\vec{c}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Similarly taking dot product from both sides of (1) by $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ we get

$$m = \frac{[\vec{c}\,\vec{a}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]} \quad \text{and} \quad n = \frac{[\vec{a}\,\vec{b}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

respectively.

$\therefore$ Then from (1) we get

$$\vec{d} = \frac{[\vec{b}\,\vec{c}\,\vec{d}]\,\vec{a} + [\vec{c}\,\vec{a}\,\vec{d}]\,\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\,\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

15(b) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ be two reciprocal system of vectors, then P.T

$$[\vec{a}\,\vec{b}\,\vec{c}][\vec{a}'\,\vec{b}'\,\vec{c}'] = 1$$

Solution: Since $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are two reciprocal system of vectors, so we get

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \text{ \& } \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Now, $[\vec{a}'\,\vec{b}'\,\vec{c}'] = \vec{a}' \cdot (\vec{b}' \times \vec{c}')$

$$= \frac{(\vec{b}\times\vec{c})}{K} \cdot \left\{ \frac{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{K^2} \right\} \quad \text{where } K = [\vec{a}\,\vec{b}\,\vec{c}]$$

$$= \frac{(\vec{b}\times\vec{c})}{K} \cdot \left\{ \frac{[\vec{c}\,\vec{a}\,\vec{b}]\,\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\,\vec{b}}{K^2} \right\}$$

$= \frac{1}{K^3}[\vec{c}\,\vec{a}\,\vec{b}]\left\{(\vec{b}\times\vec{c})\cdot\vec{a}\right\}$ $\quad [\because [\vec{c}\,\vec{a}\,\vec{a}]=0]$

$= \frac{1}{K^3}[\vec{a}\,\vec{b}\,\vec{c}]^2 = \frac{1}{K}$ $\quad (\because [\vec{a}\,\vec{b}\,\vec{c}] = K)$.

$\therefore \quad [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}']$

$= K\frac{1}{K} = 1$ (Proved).

(11) If the four vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ be such that $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$, then show that

$$\frac{|\vec{a}|}{[\vec{\beta}\,\vec{\gamma}\,\vec{\delta}]} = \frac{-|\vec{b}|}{[\vec{\gamma}\,\vec{\delta}\,\vec{\alpha}]} = \frac{|\vec{c}|}{[\vec{\delta}\,\vec{\alpha}\,\vec{\beta}]} = \frac{-|\vec{d}|}{[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}]}$$

where $\hat{\alpha}$, $\hat{\beta}$ & $\vec{\gamma}$ & $\vec{\delta}$ are unit vector along $\vec{a}$, $\vec{b}$, $\vec{c}$ & $\vec{d}$ respectively.

$\vec{\alpha}$ $\vec{a}$

$\vec{a} = |\vec{a}|\,\hat{\alpha}$

Solution: Since $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ & $\hat{\delta}$ are unit vectors along $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively, so we can write

$\vec{a} = |\vec{a}|\,\hat{\alpha}$, $\vec{b} = |\vec{b}|\,\vec{\beta}$, $\vec{c} = |\vec{c}|\,\vec{\gamma}$ &

$\vec{d} = |\vec{d}|\,\hat{\delta}$.

Since $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$, so we get

$$|\vec{a}|\,\vec{\alpha} + |\vec{b}|\,\vec{\beta} + |\vec{c}|\,\vec{\gamma} + |\vec{d}|\,\vec{\delta} = \vec{0} \quad \dots (1)$$

Taking dot product from both sides of (1) by $(\hat{\beta} \times \hat{\gamma})$ we get

$$|\vec{a}|[\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}] + |\vec{d}|[\hat{\delta}\,\hat{\beta}\,\hat{\gamma}] = 0$$

$$\Rightarrow \frac{-|\vec{d}|}{[\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]} = \frac{|\vec{a}|}{[\hat{\delta}\,\hat{\beta}\,\hat{\gamma}]} = \frac{|\vec{a}|}{[\hat{\beta}\,\hat{\gamma}\,\hat{\delta}]} \quad \cdot (2)$$

~~Similarly~~ Taking dot product ~~of~~ from both sides of (1) by $(\hat{\gamma} \times \hat{\alpha})$ and $(\hat{\alpha} \times \hat{\beta})$ we get

$$|\vec{b}|[\hat{\beta}\,\hat{\gamma}\,\hat{\alpha}] + |\vec{d}|[\hat{\delta}\,\hat{\gamma}\,\hat{\alpha}] = 0$$

$$\& \quad |\vec{c}|[\hat{\gamma}\,\hat{\alpha}\,\hat{\beta}] + |\vec{d}|[\hat{\delta}\,\hat{\alpha}\,\hat{\beta}] = 0$$

respectively.

$$\Rightarrow \frac{-|\vec{d}|}{[\hat{\beta}\,\hat{\gamma}\,\hat{\alpha}]} = \frac{|\vec{b}|}{[\hat{\delta}\,\hat{\gamma}\,\hat{\alpha}]}$$

and $$\frac{-|\vec{d}|}{[\hat{\gamma}\,\hat{\alpha}\,\hat{\beta}]} = \frac{|\vec{c}|}{[\hat{\delta}\,\hat{\alpha}\,\hat{\beta}]}$$

$$\Rightarrow \frac{-|\vec{a}|}{[\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]} = \frac{-|\vec{b}|}{[\hat{\gamma}\,\hat{\delta}\,\hat{\alpha}]} \quad \cdots (3)$$

and
$$\frac{-|\vec{a}|}{[\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]} = \frac{|\vec{c}|}{[\hat{\delta}\,\hat{\alpha}\,\hat{\beta}]} \quad \cdots (4)$$

Then from (2), (3) & (4) we get

$$\frac{|\vec{a}|}{[\hat{\beta}\,\hat{\gamma}\,\hat{\delta}]} = \frac{-|\vec{b}|}{[\hat{\gamma}\,\hat{\delta}\,\hat{\alpha}]} = \frac{|\vec{c}|}{[\hat{\delta}\,\hat{\alpha}\,\hat{\beta}]} = \frac{-|\vec{a}|}{[\hat{\alpha}\,\hat{\beta}\,\hat{\gamma}]}$$

(Proved).

15(a) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ be reciprocal system of vectors, then P.T

$$\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}(\vec{a}+\vec{b}+\vec{c})$$

Solution: Since $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors, so

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{K}, \quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{K}, \quad \vec{c}' = \frac{\vec{a}\times\vec{b}}{K}$$

where $K = [\vec{a}\,\vec{b}\,\vec{c}]$ (P.T.O).

Now, $\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}'$

$$= \frac{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})}{K^2} + \frac{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})}{K^2} + \frac{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})}{K^2}$$

$$= \frac{[\vec{b}\,\vec{c}\,\vec{a}]\vec{c} - [\vec{b}\,\vec{c}\,\vec{c}]\vec{a} + [\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b} + [\vec{a}\,\vec{b}\,\vec{c}]\vec{b} - [\vec{a}\,\vec{b}\,\vec{b}]\vec{c}}{K^2}$$

$$= \frac{[\vec{a}\,\vec{b}\,\vec{c}]\vec{c} + [\vec{a}\,\vec{b}\,\vec{c}]\vec{a} + [\vec{a}\,\vec{b}\,\vec{c}]\vec{b}}{K^2}$$

$$= \frac{1}{K}[\vec{a} + \vec{b} + \vec{c}] \qquad (\because K = [\vec{a}\,\vec{b}\,\vec{c}]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}[\vec{a} + \vec{b} + \vec{c}] \qquad \text{(Proved)}$$