**Online Course Materials :**

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**Subject : Mathematics**

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**Paper : CC-1 (Theory )**

**Unit/Chapter /Module : unit -3**

**Topic / Title : Problems and solutions on vector differentiation**

7(a) If $\vec{F} = xy\,\hat{i} - 2xy^2\,\hat{j} + zxy^3\,\hat{k}$ and $\vec{G} = 2x\,\hat{i} + y\,\hat{j} - zx^2\,\hat{k}$, then find $\dfrac{\partial^2}{\partial x \partial y}(\vec{F} \times \vec{G})$ at $(1, 1, -1)$.

Solution: Here $\vec{F} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy & -2xy^2 & zxy^3 \\ 2x & y & -zx^2 \end{vmatrix}$

$$= \left(2x^3y^2z - xy^4z\right)\hat{i}$$
$$+ \left(2x^2y^3z + x^3yz\right)\hat{j} +$$
$$\left(xy^2 + 4x^2y^2\right)\hat{k}$$

$$\therefore \frac{\partial^2}{\partial x \partial y}\left(\vec{F} \times \vec{G}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left(\vec{F} \times \vec{G}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{ \left(4x^3yz - 4xy^3z\right)\hat{i} + \left(6x^2y^2z + x^3z\right)\hat{j} + \left(2xy + 8x^2y\right)\hat{k} \right\}$$

$$= \left(12x^2yz - 4y^3z\right)\hat{i} + \left(12xy^2z + 3x^2z\right)\hat{j} + \left(2y + 16xy\right)\hat{k}$$

$$\therefore \frac{\partial^2}{\partial x \partial y}\left(\vec{F} \times \vec{G}\right)\Big|_{(1,1,-1)}$$

$$= -8\hat{i} - 15\hat{j} + 18\hat{k}$$

<u>8(a)</u> A particle moves along the curve $x = e^{-2t}$, $y = 2\cos 2t$, $z = 2\sin 3t$. Determine the velocity and acceleration at any time $t$ and their magnitudes at $t = 0$

Solution: We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

So, here $\vec{r} = e^{-2t}\,\hat{i} + 2\cos 2t\,\hat{j} + 2\sin 3t\,\hat{k}$

$\therefore$ velocity of the particle at time $t$

$$= \frac{d\vec{r}}{dt}$$

$$= -2e^{-2t}\,\hat{i} - 4\sin 2t\,\hat{j} + 6\cos 3t\,\hat{k}$$

and Acceleration of the particle at time $t$

$$= \frac{d^2\vec{r}}{dt^2}$$

$$= 4e^{-2t}\,\hat{i} - 8\cos 2t\,\hat{j} - 18\sin 3t\,\hat{k}$$

Now, Magnitude of the velocity at $t=0$

$$\left|\frac{d\vec{r}}{dt}\right|_{t=0}$$

$$= \sqrt{4e^{-4t} + 16\sin^2 2t + 36\cos^2 3t}\ \Big|_{t=0}$$

$$= \sqrt{4+36} = \sqrt{40}$$

Magnitude of the acceleration at $t=0$ is $\left|\frac{d^2\vec{r}}{dt^2}\right|_{t=0}$

$$= \sqrt{16e^{-4t} + 64\sin^2 2t + (18)^2\sin^2 3t}\ \Big|_{t=0}$$

$$= \sqrt{16 + \cancel{324}\,64} = \sqrt{\cancel{340}\ 80}$$

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(9) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. Find the components of velocity and acceleration at time $t = 1$ in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$

<u>Solution</u> : We know that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t - 5)\hat{k}$$

$\therefore \frac{d\vec{r}}{dt}$ = velocity of the particle.

$$= 4t\,\hat{i} + (2t - 4)\hat{j} + 3\hat{k}$$

& $\frac{d^2\vec{r}}{dt^2}$ = Acceleration of the particle

$$= 4\hat{i} + 2\hat{j}.$$

Velocity of the particle at $t = 1$

$$= \left.\frac{d\vec{r}}{dt}\right|_{t=1} = 4\hat{i} - 2\hat{j} + 3\hat{k}$$

$\therefore$ Component of the velocity of the particle at $t = 1$ in the direction of $(\hat{i} - 3\hat{j} + 2\hat{k})$ is

$$\left\{\frac{(4\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} + 2\hat{k})}{|\hat{i} + 3\hat{j} + 2\hat{k}|}\right\} \frac{(\hat{i} - 3\hat{j} + 2\hat{k})}{|\hat{i} - 3\hat{j} + 2\hat{k}|}$$

ie $\left(\frac{4 + 6 + 6}{1 + 9 + 4}\right)(\hat{i} - 3\hat{j} + 2\hat{k})$

ie $\frac{8}{7}(\hat{i} - 3\hat{j} + 2\hat{k})$

Again the component of the acceleration of the particle at $t = 1$ in the direction of $(\hat{i} - 3\hat{j} + 2\hat{k})$ is $\left(\frac{(4\hat{i} + 2\hat{j}) \cdot (\hat{i} - 3\hat{j} + 2\hat{k})}{|\hat{i} - 3\hat{j} + 2\hat{k}|}\right) \frac{\hat{i} - 3\hat{j} + 2\hat{k}}{|\hat{i} - 3\hat{j} + 2\hat{k}|}$

$$= \left(\frac{4-6}{14}\right)\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$$

$$= -\frac{1}{7}\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$$

(7) Show that $\dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$ is same at all points of the curve whose vector equ$^n$ is $\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 2t\,\hat{k}$

Solution: Here $\vec{r} = 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 2t\,\hat{k}$ --- (1)

$\therefore \dot{\vec{r}} = -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 2\hat{k}$

$\ddot{\vec{r}} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$

$$\therefore \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\sin t & 4\cos t & 2 \\ -4\cos t & -4\sin t & 0 \end{vmatrix}$$

$$= 8\sin t\,\hat{i} - 8\cos t\,\hat{j} + 16\hat{k}$$

$$\therefore \quad \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

$$= \frac{\sqrt{64\sin^2 t + 64\cos^2 t + (16)^2}}{\sqrt{16\sin^2 t + 16\cos^2 t + 4}}$$

$$= \frac{\sqrt{64+256}}{\left(\sqrt{16+4}\right)^3} = \frac{1}{20}\sqrt{\frac{320}{20}}$$

$$= \frac{1}{20}\sqrt{16} = \frac{1}{5},$$

a constant not depending on $t$

$\therefore \quad \dfrac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$ is same at all points of the given curve (Proved).

(8) Prove that if $\vec{F}$ & $\vec{G}$ are both differentiable vector function of a scalar variable $t$, then $\vec{F} \times \vec{G}$ is a differentiable vector function of $t$ and $\frac{d}{dt}(\vec{F} \times \vec{G}) = \frac{d\vec{F}}{dt} \times \vec{G} + \vec{F} \times \frac{d\vec{G}}{dt}$

Solution: Since $\vec{F}$ and $\vec{G}$ both are differentiable function, so

$$\frac{d\vec{F}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} \text{ \& } \frac{d\vec{G}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{G}(t+\Delta t) - \vec{G}(t)}{\Delta t}$$

both exist finitely ....(1)

Let $\vec{h}(t) = \vec{F}(t) \times \vec{G}(t)$

Now, $$\underset{\Delta t \to 0}{Lt} \frac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{Lt} \frac{\vec{F}(t+\Delta t) \times \vec{G}(t+\Delta t) - \vec{F}(t) \times \vec{G}(t)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{Lt} \frac{\left\{ \vec{F}(t+\Delta t) - \vec{F}(t) \right] \times \vec{G}(t+\Delta t) + \vec{F}(t) \times \left( \vec{G}(t+\Delta t) - \vec{G}(t) \right)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{Lt} \left\{ \left( \frac{\vec{F}(t+\Delta t) - \vec{F}(t)}{\Delta t} \right) \times \vec{G}(t+\Delta t) + \vec{F}(t) \times \left( \frac{\vec{G}(t+\Delta t) - \vec{G}(t)}{\Delta t} \right) \right\}$$

$$= \frac{d\vec{F}}{dt} \times \vec{G}(t) + \vec{F}(t) \times \frac{d\vec{G}}{dt}$$

(by (1))

Since $\underset{\Delta t \to 0}{Lt} \dfrac{\vec{h}(t+\Delta t) - \vec{h}(t)}{\Delta t}$ exists finitely, so $\vec{h}(t)$ ie $\vec{F} \times \vec{G}$ is differentiable and

$$\frac{d\vec{h}}{dt} = \frac{d}{dt}\left(\vec{F} \times \vec{G}\right) = \frac{d\vec{F}}{dt} \times \vec{G} + \vec{F} \times \frac{d\vec{G}}{dt}$$