Online Course Materials :

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Topic / Title : Problems and solutions on vector equations

13 Solve the eqn $\vec{r} \times \vec{p} = \vec{q}$ where $\vec{p}\,(\neq \vec{0})$ and $\vec{q}$ are given vectors such that $\vec{p} \cdot \vec{q} = 0$.

Solution: Let $\vec{r} \times \vec{p} = \vec{q}$ ....(1)

Since $\vec{p}$, $\vec{q}$ and $\vec{p} \times \vec{q}$ are non coplanar vectors, so any vector $\vec{r}$ can be expressed as

$$\vec{r} = c_1 \vec{p} + c_2 \vec{q} + c_3 (\vec{p} \times \vec{q}) \quad ...(2)$$

Now $\vec{r}$ will be a solution of (1) if

$$\left\{ c_1 \vec{p} + c_2 \vec{q} + c_3 (\vec{p} \times \vec{q}) \right\} \times \vec{p} = \vec{q}$$

$$\Rightarrow c_2 (\vec{q} \times \vec{p}) + c_3 (\vec{p} \times \vec{q}) \times \vec{p} = \vec{q}$$

$$\Rightarrow c_2 (\vec{q} \times \vec{p}) + c_3 \left\{ (\vec{p} \cdot \vec{p})\vec{q} - (\vec{q} \cdot \vec{p})\vec{p} \right\} = \vec{q}$$

or $c_2(\vec{q}\times\vec{p}) + c_3|\vec{p}|^2\vec{q} = \vec{q}$ $\quad(\because \vec{p}\cdot\vec{q}=0)$

or $c_2(\vec{q}\times\vec{p}) + \left(c_3|\vec{p}|^2 - 1\right)\vec{q} = \vec{0}$

$\Rightarrow$ $c_2 = 0$ and $c_3|\vec{p}|^2 - 1 = 0$

(as $\vec{p}$ and $\vec{q}\times\vec{p}$ are not parallel vectors)

$$\Rightarrow \quad c_2 = 0 \quad \& \quad c_3 = \frac{1}{|\vec{p}|^2}$$

Then from (2) we get

$$\vec{r} = c_1\vec{p} + \frac{1}{|\vec{p}|^2}(\vec{p}\times\vec{q})$$

which is the solution of (1) where c_1 is an arbitrary constant.

14(b) Solve the following for $\vec{x}$ where K is a non zero given scalar and $\vec{a}, \vec{b}$ are two given non zero vectors: $K\vec{x} + \vec{x} \times \vec{a} = \vec{b}$

Solution: Let $K\vec{x} + \vec{x} \times \vec{a} = \vec{b}$ ...(1)

Since $\vec{a}$ & $\vec{b}$ are non zero vectors, so $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ are non coplanar vectors.

Let $\vec{x} = c_1\vec{a} + c_2\vec{b} + c_3(\vec{a} \times \vec{b})$ ...(2)

Then $\vec{x}$ will be a solution of (1) if

$$K\{c_1\vec{a} + c_2\vec{b} + c_3(\vec{a}\times\vec{b})\} + \{c_1\vec{a} + c_2\vec{b} + c_3(\vec{a}\times\vec{b})\}\times\vec{a} = \vec{b}$$

or $$K\{c_1\vec{a} + c_2\vec{b} + c_3(\vec{a}\times\vec{b})\} + c_2\,\vec{b}\times\vec{a} + c_3\{(\vec{a}\times\vec{b})\times\vec{a}\} = \vec{b}$$

or $$K\{c_1\vec{a} + c_2\vec{b} + c_3(\vec{a}\times\vec{b})\} + c_2\,\vec{b}\times\vec{a} + c_3\{|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\} = \vec{b}$$

or $$Kc_1\vec{a} + Kc_2\vec{b} + Kc_3(\vec{a}\times\vec{b}) + c_2\,\vec{b}\times\vec{a} + c_3\{|\vec{a}|^2\vec{b} - (\vec{a}\cdot\vec{b})\vec{a}\} = \vec{b}$$

or $$\{Kc_1 - c_3(\vec{a}\cdot\vec{b})\}\vec{a} + (Kc_2 + c_3|\vec{a}|^2 - 1)\vec{b} + (Kc_3 - c_2)(\vec{a}\times\vec{b}) = \vec{0}$$

$\therefore \quad K c_1 - c_3 (\vec{a}\cdot\vec{b}) = 0 \quad \cdots (3)$

$K c_2 + c_3 |\vec{a}|^2 - 1 = 0 \quad \cdots (4)$

$K c_3 - c_2 = 0 \quad \cdots (5)$

from (5), $c_3 = \frac{1}{K} c_2$

Then from (4) we get $K c_2 + \frac{1}{K} c_2 |\vec{a}|^2 - 1 = 0$

or $c_2 \left(\frac{K^2 + |\vec{a}|^2}{K} \right) = 1$

or $c_2 = \frac{K}{K^2 + |\vec{a}|^2}$

Then $c_3 = \frac{1}{K^2 + |\vec{a}|^2}$

Then from (3), $c_1 = \frac{\vec{a}\cdot\vec{b}}{K(K^2 + |\vec{a}|^2)}$

$$\therefore \quad \vec{x} = \frac{\vec{a}\cdot\vec{b}}{K(K^2 + |\vec{a}|^2)} \vec{a} + \frac{K}{K^2 + |\vec{a}|^2} \vec{b} + \frac{1}{K^2 + |\vec{a}|^2} (\vec{a} \times \vec{b})$$

$$= \frac{1}{K^2 + |\vec{a}|^2} \left\{ \frac{\vec{a}\cdot\vec{b}}{K} \vec{a} + K \vec{b} + \vec{a} \times \vec{b} \right\}$$

which is the solution of (1)

(13) Show that a necessary and sufficient condition that the vector equⁿ. $\vec{a} \times \vec{x} = \vec{b}$, where $\vec{a}$ & $\vec{b}$ are given vectors and $\vec{a}$ is a non zero vector, possesses a solution is that $\vec{a} \cdot \vec{b} = 0$

<u>solution:</u> Let $\vec{a} \times \vec{x} = \vec{b}$ - - - (1)

Let solution of (1) be exist. Let $\vec{x} = \vec{p}$ be a solution of (1). Then $\vec{a} \times \vec{p} = \vec{b}$

Now, $\vec{a} \cdot \vec{b} = \vec{a} \cdot (\vec{a} \times \vec{p})$ $(\because \vec{b} = \vec{a} \times \vec{p})$

$= [\vec{a}\, \vec{a}\, \vec{p}] = 0$

which is one part.

Let $\vec{a} \cdot \vec{b} = 0$. Clearly $\vec{a}$, $\vec{b}$ and $\vec{a} \times \vec{b}$ are non coplanar vectors

Let $\vec{x} = c_1 \vec{a} + c_2 \vec{b} + c_3 (\vec{a} \times \vec{b})$

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Now $\vec{a} \times \left(c_1 \vec{a} + c_2 \vec{b} + c_3\, \vec{a} \times \vec{b} \right)$

$$= c_1(\vec{a} \times \vec{a}) + c_2(\vec{a} \times \vec{b}) + c_3\, \vec{a} \times (\vec{a} \times \vec{b})$$

$$= c_2(\vec{a} \times \vec{b}) + c_3 \left\{ (\vec{a} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{a})\vec{b} \right\}$$

$$= c_2(\vec{a} \times \vec{b}) - c_3 |\vec{a}|^2 \vec{b} \quad (\because \vec{a} \cdot \vec{b} = 0)$$

$$= \vec{b} \quad \text{if } c_2 = 0 \ \& \ c_3 = -\frac{1}{|\vec{a}|^2}$$

$\therefore$ $\vec{x} = c_1 \vec{a} - \dfrac{1}{|\vec{a}|^2} (\vec{a} \times \vec{b})$ satisfies the eqn $\vec{a} \times \vec{x} = \vec{b}$ and hence the eqn $\vec{a} \times \vec{x} = \vec{b}$ has solution.