Online Course Materials :

Prepared by: Masiur Rahaman Sardar

Assistant Professor

Dept. Of Mathematics

City College

Mobile : 9830458374

Email - sardarmasiur@gmail.com

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Subject: Vector operator & differentiation

8(b) differentiate w.r.t t

$$r^2 \vec{r} + (\vec{a}\cdot\vec{r})\vec{b}$$ where

$|\vec{r}| = r$ & $\vec{a}, \vec{b}$ are constant

vectors.

Solution:

$$\frac{d}{dt}\left\{ r^2 \vec{r} + (\vec{a}\cdot\vec{r})\vec{b} \right\}$$

$$= \frac{d}{dt}(r^2)\vec{r} + r^2 \frac{d\vec{r}}{dt} +$$

$$\left\{\frac{d}{dt}(\vec{a}\cdot\vec{r})\right\}\vec{b} + (\vec{a}\cdot\vec{r})\frac{d\vec{b}}{dt}$$

$$= 2r\frac{dr}{dt}\vec{r} + r^2\frac{d\vec{r}}{dt} + \left\{\left(\frac{d\vec{a}}{dt}\cdot\vec{r}\right) + \left(\vec{a}\cdot\frac{d\vec{r}}{dt}\right)\right\}\vec{b}$$

$$+ \vec{0} \quad \left(\because \frac{d\vec{b}}{dt} = 0\right)$$

$$= 2r\frac{dr}{dt}\vec{r} + r^2\frac{d\vec{r}}{dt} + \left(\vec{a}\cdot\frac{d\vec{r}}{dt}\right)\vec{b}$$

$\left(\because \frac{d\vec{a}}{dt} = \vec{0}\right.$ as $\vec{a}$ is a constant vector $\left.\right)$.

(10) Show that $\vec{A} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}$ is irrotational. Find ϕ such that $\vec{A} = \vec{\nabla}\phi$ and $\phi(1,1,1) = 3$

Solution: $\operatorname{curl}\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$

$$= (-1+1)\hat{i} + (3z^2 - 3z^2)\hat{j} + (6x - 6x)\hat{k}$$

$$= \vec{0}$$

$\therefore \vec{A}$ is an irrotational vector.

2nd part : Let $\vec{A} = \vec{\nabla}\phi$

$$= \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial \phi}{\partial x} = 6xy + z^3$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y$$

We Know that

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= (6xy + z^3)dx + (3x^2 - z)dy + (3xz^2 - y)dz$$

$= 6xy\,dx + 3x^2 dy + (z^3 dx + 3x z^2 dz)$
$- (z\,dy + y\,dz)$

$= d(3x^2 y) + d(z^3 x) - d(yz)$

$= d(3x^2 y + z^3 x - yz)$

Integrating,

$\phi = 3x^2 y + z^3 x - yz + c$ where c is an arbitrary constant.

Since $\phi(1,1,1) = 3$, so we get

$3 + 1 - 1 + c = 3$

$\Rightarrow c = 0$

$\therefore \phi = 3x^2 y + z^3 x - yz$.

Alternatively: Since $\dfrac{\partial \phi}{\partial x} = 6xy + z^3$, so

$\phi = 3x^2 y + z^3 x + \psi_1(y, z)$ ...(1)

Again as $\frac{\partial \varphi}{\partial y} = 3x^2 - z$, so we get

$$\varphi = 3x^2y - yz + \psi_2(x, z) \quad \cdots (2)$$

Again $\frac{\partial \varphi}{\partial z} = 3xz^2 - y$, so we get

$$\varphi = xz^3 - yz + \psi_3(x, y) \quad \cdots (3)$$

From (1), (2) & (3), we get that

$$\varphi = 3x^2y + z^3x - yz + c$$ where

c is an arbitrary constant.

Since $\varphi(1,1,1) = 3$, so $c = 0$

$$\therefore \varphi = 3x^2y + z^3x - yz.$$

(11) Prove that $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$

Solution: Let $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

Then $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\therefore \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \sum \left(\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x \partial z}\right)\hat{i} \quad \cdots (1)$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}$$

$$= \vec{\nabla}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\hat{i} - \sum\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$= \sum\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$= \sum\left(\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i} \quad \ldots(2)$$

from (1) & (2) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$ (proved).

10. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $\vec{a}$ is a constant vector, then show that

a) curl $(\vec{a} \times \vec{r}) = 2\vec{a}$

b) grad $r^{n} = n r^{n-2}\vec{r}$

c) curl $\left\{ r^{n} (\vec{a} \times \vec{r}) \right\} = (n+2) r^{n} \vec{a} - n r^{n-2} (\vec{r} \cdot \vec{a}) \vec{r}$

Solution: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2 & a_3 are scalars.

Then
$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= (a_2 z - a_3 y)\hat{i} + (a_3 x - a_1 z)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\therefore \text{curl} \left\{ r^{n} (\vec{a} \times \vec{r}) \right\} = \vec{\nabla} \times \left\{ r^{n} (\vec{a} \times \vec{r}) \right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n(a_2 z - a_3 y) & r^n(a_3 x - a_1 z) & r^n(a_1 y - a_2 x) \end{vmatrix}$$

$$= \sum \left\{ \frac{\partial}{\partial y}\left\{ r^n (a_1 y - a_2 x) \right\} - \frac{\partial}{\partial z}\left(r^n (a_3 x - a_1 z)\right) \right\}$$

$$= \sum \left\{ \left(r^n a_1 + n r^{n-1} \frac{\partial r}{\partial y} (a_1 y - a_2 x) \right) - \left\{ r^n(-a_1) + (a_3 x - a_1 z)\, n r^{n-1} \frac{\partial r}{\partial z} \right\} \right\} \hat{i}$$

$$= \sum \left\{ r^n a_1 + n r^{n-1} \frac{y}{r}(a_1 y - a_2 x) + a_1 r^n - n r^{n-1} \frac{z}{r} (a_3 x - a_1 z) \right\} \hat{i}$$

$$\left[\because r^2 = x^2 + y^2 + z^2 \quad 2r \frac{\partial r}{\partial y} = 2y \right]$$

$$= \sum \left[\left(2 r^n a_1 + n r^{n-2} \left(y (a_1 y - a_2 x) - z (a_3 x - a_1 z) \right) \right) \right] \hat{i}$$

$$= 2 r^n \sum a_i \hat{i} + n r^{n-2} \Big[\left\{ y(a_1 y - a_2 x) - z(a_3 x - a_1 z) \right\} \hat{i} + \left(z(a_2 z - a_3 y) - x(a_1 y - a_2 x) \right) \hat{j} + \left(x(a_3 x - a_1 z) - y(a_2 z - a_3 y) \right) \hat{k} \Big]$$

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$$= 2r^n \vec{a} + n r^{n-2} \Big\{ \cancel{a_1 y^2 \hat{i} + a_1 z^2} \; y^2 (a_1 \hat{i} + a_3 \hat{k}) + x^2 (a_2 \hat{j} + a_3 \hat{k}) + z^2 (a_1 \hat{i} + a_2 \hat{j}) - a_2 xy \hat{i} - a_3 xz \hat{i} - a_3 zy \hat{j} - a_1 xy \hat{j} - a_1 xz \hat{k} - a_2 yz \hat{k} \Big\}$$

$$= 2r^n \vec{a} + n r^{n-2} \Big\{ y^2 \vec{a} - y^2 a_2 \hat{j} + x^2 \vec{a} - x^2 a_1 \hat{i} + z^2 \vec{a} - z^2 a_3 \hat{k} - a_2 xy \hat{i} - a_3 xz \hat{i} - a_3 zy \hat{j} - a_1 xy \hat{j} - a_1 xz \hat{k} - a_2 yz \hat{k} \Big\}$$

$$= 2r^n \vec{a} + n r^{n-2} \Big\{ (x^2+y^2+z^2)\vec{a} - y a_2 (y\hat{j} + x\hat{i} + z\hat{k}) - x a_1 (x\hat{i} + y\hat{j} + z\hat{k}) - z a_3 (z\hat{k} + y\hat{j} + x\hat{i}) \Big\}$$

$$= 2r^n \vec{a} + n r^{n-2} \Big\{ r^2 \vec{a} - (x a_1 + y a_2 + z a_3)\vec{r} \Big\}$$

$$= 2r^n \vec{a} + n r^n \vec{a} - n r^{n-2} (\vec{r} \cdot \vec{a}) \vec{r} = (2+n) r^n \vec{a} - n r^{n-2} (\vec{r} \cdot \vec{a}) \vec{r}$$

If $\hat{a}$ is a unit vector in the direction of $\vec{b}$, show that $\hat{a} \times \frac{d\hat{a}}{dt} = \frac{1}{b^2}\left(\vec{b} \times \frac{d\vec{b}}{dt}\right)$

Solution: Since $\hat{a}$ is a unit vector in the direction of $\vec{b}$, so we get

$$\hat{a} = \frac{\vec{b}}{b} \quad \text{where } b = |\vec{b}|$$

Now

$$\hat{a} \times \frac{d\hat{a}}{dt} = \frac{\vec{b}}{b} \times \frac{d}{dt}\left(\frac{\vec{b}}{b}\right)$$

$$= \frac{\vec{b}}{b} \times \left(\frac{1}{b}\frac{d\vec{b}}{dt} + \left(-\frac{1}{b^2}\frac{db}{dt}\vec{b}\right)\right.$$

$$= \frac{1}{b^2}\left(\vec{b} \times \frac{d\vec{b}}{dt}\right) - \frac{1}{b^3}\frac{db}{dt}\left(\vec{b} \times \vec{b}\right)$$

$$= \frac{1}{b^2}\left(\vec{b} \times \frac{d\vec{b}}{dt}\right)$$

11(a) Find the directional derivatives of the function $f(x,y,z) = x^2 - y^2 + z^2$ at the pt $P(1,2,-3)$ in the direction of the vector $\overrightarrow{PQ}$ where Q is the pt $(3,1,2)$ [2]

Solution: Let $\hat{n}$ be the unit vector in the direction of $\overrightarrow{PQ}$

Then $\hat{n} = \dfrac{2\hat{i} - \hat{j} + 5\hat{k}}{\sqrt{4+1+25}}$

$\therefore$ The directional derivatives of $f(x,y,z)$ at $(1,2,-3)$ $= (\vec{\nabla} f) \cdot \hat{n} \Big|_{(1,2,-3)}$

$= \left(2x\hat{i} - 2y\hat{j} + 2z\hat{k}\right) \cdot \left(\dfrac{2\hat{i} - \hat{j} + 5\hat{k}}{\sqrt{30}}\right)\Big|_{(1,2,-3)}$

$$= \frac{2}{\sqrt{30}}\left(2x + y + 5z\right)\Big|_{(1,2,-3)}$$

$$= \frac{2}{\sqrt{30}}\left(2 + 2 - 15\right)$$

$$= -\frac{22}{\sqrt{30}}$$

10. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $f(r)$ be a scalar function possessing first and 2nd order derivatives. Prove that

$$\nabla^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

Solution: Since $r = |\vec{r}|$, so $r^2 = x^2 + y^2 + z^2$

Now $$\frac{\partial}{\partial x}\left(f(r)\right) = \frac{\partial}{\partial r}\left(f(r)\right)\frac{\partial r}{\partial x}$$

$$= \frac{df}{dr}\,\frac{x}{r} \quad \left(\because r^2 = x^2 + y^2 + z^2, \text{ so } 2r\frac{\partial r}{\partial x} = 2x \right.$$

Similarly $$\frac{\partial f}{\partial y} = \frac{df}{dr}\,\frac{y}{r}, \quad \frac{\partial f}{\partial z} = \frac{df}{dr}\,\frac{z}{r}$$

Now, $\nabla^2 f(r) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dr}\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dr}\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{df}{dr}\frac{z}{r}\right)$$

$$= \frac{df}{dr}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial x}\left(\frac{df}{dr}\right)\frac{x}{r} + \cdots\cdots$$

$$= \frac{df}{dr}\left(\frac{r.1 - x\frac{\partial r}{\partial x}}{r^2}\right) + \frac{\partial}{\partial r}\left(\frac{df}{dr}\right)\frac{\partial r}{\partial x}\frac{x}{r} + \cdots\cdots$$

$$= \frac{df}{dr}\frac{r - \frac{x^2}{r}}{r^2} + \frac{d^2 f}{dr^2}\frac{x^2}{r^2} + \frac{df}{dr}\frac{r - \frac{y^2}{r}}{r^2}$$

$$+ \frac{d^2 f}{dr^2}\left(\frac{y}{r}\right)^2 + \frac{df}{dr}\frac{r - \frac{z^2}{r}}{r^2} + \frac{d^2 f}{dr^2}\left(\frac{z}{r}\right)^2$$

$$= \frac{d^2 f}{dr^2}\left(\frac{x^2+y^2+z^2}{r^2}\right) + \frac{df}{dr}\left(\frac{3r^2 - (x^2+y^2+z^2)}{r^3}\right)$$

$$= \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$ (Proved).

11 (a) Show that the vector field determined by vector $r^n \vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$ is solenoidal for $n = -3$

Solution: Since $n = -3$, so $r^n \vec{r} = \dfrac{\vec{r}}{r^3}$

Now $\operatorname{div}\left(\dfrac{\vec{r}}{r^3}\right)$

$$= \operatorname{div}\left(\frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)$$

$$= \sum \frac{r^3 \cdot 1 - x\, 3r^2 \frac{\partial r}{\partial x}}{r^6}$$

$$= \sum \frac{r^3 - 3r^2 \frac{x^2}{r}}{r^6} \qquad \left(\because r^2 = x^2+y^2+z^2,\ \therefore 2r\frac{\partial r}{\partial x} = 2x\right)$$

$$= \sum \frac{r^3 - 3rx^2}{r^6}$$

$$= \frac{3r^3 - 3r(x^2+y^2+z^2)}{r^6}$$

$$= \frac{3r^3 - 3r^3}{r^6} = 0$$

$\therefore \frac{\vec{r}}{r^3}$ is a solenoidal vector

(b) For what value of the constant a, will the vector field $\vec{A} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ be always irrotational?

<u>Solution:</u> Since $\bar{A}$ is irrotational, so curl $\vec{A} = \vec{0}$

Then
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \left(-3z^2 - (1-a)z^2\right)\hat{j} + \left(2x(a-2) - ax\right)\hat{k} = \vec{0}$$

$\Rightarrow -3 - (1-a) = 0 \quad \ldots \qquad \Rightarrow a - 4 = 0$

$a - 4 = 0 \qquad\qquad \Rightarrow a = 4$ (Ans).

9. Define directional derivatives of a scalar pt function ϕ at a pt along any semi line through that pt. Find the maximum value of the directional derivatives of ϕ $= xy^2 + 2yz - 3x^3z^2$ at the pt $(1,-1,1)$. Find also the directional in which it occurs. [2+2+1].

Solution:

We consider a pt P and a semi line $\vec{PQ}$ through P. Then the directional derivative of ϕ at P along $\vec{PQ}$ is defined by

$$(\vec{\nabla}\phi)_P \cdot \frac{\vec{PQ}}{|\vec{PQ}|}$$

where $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$.

2nd part : $\vec{\nabla}\phi = (y^2 - 9x^2z^2)\hat{i} + (2xy + 2z)\hat{j}$
$(2y - 6x^3z)\hat{k}$

$\therefore \vec{\nabla}\phi \Big|_{(1,-1,1)} = -8\hat{i} - 8\hat{k}$

$\therefore$ Maximum directional derivative of ϕ

$= |\vec{\nabla}\phi| = \sqrt{64 + 64} = 8\sqrt{2}$

It occurs in the direction of $(\vec{\nabla}\phi)_{(1,-1,1)}$

i.e. in the direction of $-8(\hat{i} + \hat{k})$